N.G. Chetaev

The Stability of Motion

THE STABILITY OF MOTION

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from the Russian by

N. G. CHETAYEV

Translated by

MORTON NADLER

Translation Editors

A. W. BABISTER
Senior Lecturer in Aerodynamics

J. BURLAK
Lecturer in Mathematics

University of Glasgow

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FOREWORD

Questions of stability are important in both theory and practice. The stability problems which are becoming more and more frequent in engineering and physics require Lyapunov's exact methods, because more approximate methods are unsatisfactory. So a study of Lyapunov's methods and their application is becoming increasingly necessary for ordinary engineering practice.

From this arises one of the basic aims of this book, in which I present those investigations of Lyapunov's which appear to me to have the most bearing on applied problems in the stability of motion. However I have not attempted in this book to make a complete presentation of all the results obtained.

I have tried to present Lyapunov's results in a simple manner, without false modernizations; I have taken the information required from the theory of systems of linear differential equations with constant coefficients, and presented certain of the results. Some of the examples (on the longitudinal stability of aircraft, etc.) proceed from the familiar modelling of systems with an infinite number of degrees of freedom, and are presented without discussion of the validity or otherwise of the model.

Where, in this book, references are made to Lyapunov's monograph on the general problem of stability of motion, the paragraph number in the text is followed by square brackets containing the number of the corresponding paragraph in the monograph and the letter "L".

In this, the second edition, certain examples have been replaced by others.

Advances achieved in the theory of stability of motion since the first edition of the book have not been included in the present edition. To describe them completely would require many volumes; they are not fully treated even in

the fine books of Nemytskii and Stepanov[1], Malkin[2], Lur'e[3], Duboshin[4], Aizerman[5], Gantmakher[6], and Letov[7]. The investigations by Bogolyubov, Persidskii, Erugin, Krein, Barbashin, Krasovskii, and others deserve independent books, written in the styles of the individual authors.

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CHAPTER 1

THE PROBLEM OF STABILITY

Introductory Remarks

1. Dynamics is the science of the virtual equilibria and motions of mechanical systems. Galileo and Newton discovered its principles and proved their validity by experiments on falling bodies and by explanation of the motions of the planets. But not every case of a mechanical system corresponding to a mathematically rigorous solution of either the equations of equilibrium or the differential equations of motion can, in fact, be observed. (For example, no-one has seen a pencil stand vertically on its sharpened end on a smooth horizontal table). This is because there are, in the initial state of the mechanical system, small forces and deviations which have to be ignored, even in rigorous theory but which undeniably exist in reality and disturb the equilibrium and motion - in some cases only slightly but in others strongly. Equilibria and motions which are only slightly disturbed by these apparently negligible deviations are termed stable; those which suffer extensive disturbance are termed unstable.

A general principle for selecting solutions corresponding to stable states has not been given in mechanics, which has taken on the character of a science of idealized systems. For its rigorous application to nature, the solution of the problem of stability has to be considered afresh for each system. Only Torricelli, when discussing the theory of the statics of solid bodies as existed in his time, proposed a principle (the origin of which is lost in the remote past), which gave only stable positions of equilibrium. He postulated that "Two bodies, which are connected in some manner, cannot move of their own accord unless their centre of gravity can fall".

The formulation of Torricelli's principle arose from

another principle which provided an easy solution of various questions of statics. This principle is as follows: In a system of solid bodies in a state of equilibrium, the centre of gravity occupies the lowest relative position possible[1].

There is no corresponding dynamic principle for the selection of rigorous solutions for stable motion, although stability problems have occupied the attention of many eminent mathematicians - Lagrange, Kelvin, Routh, Zhukovskii, and Poincaré. Lagrange generalized Torricelli's principle, proving the theorem on the stability of an isolated equilibrium of a mechanical system when the force function of the forces acting on the system has a maximum in this equilibrium position. Routh, developing the method of ignoring cyclical co-ordinates, by simple extension of this theorem of Lagrange, found a stability criterion for certain cyclical motions.

The general problem on stability of motion in its classical formulation was solved by Lyapunov in his celebrated monograph "The General Problem Of Stability Of Motion" (Khar'kov, 1892)[2].

2. Engineering also poses problems as to the feasibility of projects based, in essence, on some solution of either an equilibrium equation or differential equations of motion of a proposed mechanical system. The necessity of solving these problems inevitably leads the engineer to problems of stability, since the realization of a project is accompanied by certain production tolerances; from engineering considerations themselves the system must operate under the action of forces not fully considered in the project.

For example, if a passenger aircraft is designed, it is necessary to guarantee a certain type of stability in its projected motions, so as to obtain a machine which is steady in flight and safe in take-off and landing. A crankshaft must be designed so as not to break under vibrations which may arise under actual operating conditions of a motor. To ensure that guns possess the greatest accuracy and precision of fire, it is necessary to design the gun and shells to have a certain type of stability of trajectory.

There are many such examples, but they add nothing to the fact that in solving problems of the stability of motion it is necessary to concentrate on those solutions of the equations of mechanics which correspond to conditions of

stability. In those cases where it is desirable to avoid a particular solution it is reasonable, by some change in the design of a mechanical system, to make the state of motion corresponding to this solution unstable.

Statement of the Problem

3[1L]. Let us consider some holonomic mechanical system. Let $q_1, \ldots, q_k$ denote its independent Lagrangian co-ordinates and $q_1', \ldots, q_k'$ the generalized velocities. In the dynamic problem, when the forces are given in a defined manner, the variables q_j satisfy k ordinary differential equations of the second order.

To a particular solution of these equations:

$$q_j = f_j(t) \qquad (j = 1, \ldots, k)$$

there corresponds a certain definite motion of the system. Comparing it in certain respects with other possible motions under the same forces, this motion will be termed unperturbed, while the others with which it is compared will be termed perturbed.

Let t_0 be the moment of time arbitrarily taken as the origin, while q_{j0} and q_{j0}' denote the initial values of the variables q_j and of their derivatives with respect to time q_j' For the unperturbed motion:

$$q_{j0} = f_j(t_0), \qquad q_{j0}' = f_j'(t_0),$$

and, for the perturbed motion:

$$q_{j0} = f_j(t_0) + \varepsilon_j, \qquad q_{j0}' = f_j'(t_0) + \varepsilon_j',$$

where ε_j, ε_j' are certain real constants, which we shall term perturbations. With the assignment of these constants the perturbed motion is completely defined, since the forces acting on the system are assumed invariant.

In referring to a perturbed motion being close to the unperturbed motion, we shall mean a motion for which the perturbations are sufficiently small numerically.

At the same time, for a perturbed motion which is close to the unperturbed motion, the differences between the values of the co-ordinates q_j and the velocities q_j' in these

two motions, small by definition at the initial instant, (and, by continuity, for values of t differing little from t_0) need not be small for instants of time sufficiently remote from the origin.

The deviation of a perturbed motion from the unperturbed motion may, in fact, be noticed through differences in the values of certain quantities measured in the observations or experiment, dependent on the motion. Not wishing to omit cases where quantities are observed which differ from the co-ordinates q_j and the velocities q_j', let us consider certain given continuous real functions $Q_1, \ldots, Q_n$ of the quantities q_j, q_j' and the time t.

For an unperturbed motion the functions Q_s, after the substitutions $q_j = f_j(t)$ and $q_j' = f_j'(t)$, become certain known functions of t which are denoted by $F_1, \ldots, F_n$ respectively, and for a perturbed motion (retaining for them the previous notation Q_s) they will be certain functions of the time t and the perturbations ε_j, ε_j'.

When all the perturbations ε_j, ε_j' vanish, the difference

$$x_s = Q_s - F_s$$

will also vanish for all values of t. But if the non-vanishing perturbations ε_j, ε_j' are all assumed to be arbitrarily small, the question arises whether it is possible to assign arbitrarily small constants, which will never be exceeded by the absolute magnitudes of the differences x_s. We shall deal only with those cases where the solution of this problem is independent of the choice of the moment t_0 in which the perturbation is communicated. We shall adopt the following definitions of Lyapunov.

Let $L_1, \ldots, L_n$ be arbitrarily given positive numbers. If for all small arbitrary values of L_s there may be chosen positive numbers $E_1, \ldots, E_k$ and $E_1', \ldots, E_k'$, such that for all perturbations ε_j, ε_j' satisfying the conditions:

$$|\varepsilon_j| \leqslant E_j, \quad |\varepsilon_j'| \leqslant E_j',$$

and for all values of t exceeding t_0, the inequalities:

$$|Q_s - F_s| < L_s,$$

are satisfied, then the unperturbed motion is stable with

respect to the quantities $Q_1, \ldots, Q_n$; in the opposite case it is unstable.

It may be that limits E_j, E_j' which satisfy this definition cannot be found if all perturbations are considered, and that such limits may be found for perturbations subject to certain conditions of the form

$$f = 0 \quad \text{or} \quad f > 0,$$

where f is some function of the perturbations ε_j, ε_j' which vanishes when all perturbations are assumed equal to zero. In such cases we shall say that the unperturbed motion is stable for perturbations subject to the given conditions.

In the definition of stability, Lyapunov utilized the concept of number and not of infinitely small quantity. Therefore, according to Lyapunov, the essence of the stability concept lies not so much in the character of variation of the quantities $|Q_s - F_s|$ when the perturbations ε_j, ε_j' tend to zero, as in estimations of the numerical values of perturbation for given numerical estimates of the differences $|Q_s - F_s|$ for an unperturbed motion stable with regard to the functions Q_s. Consequently, it cannot be stated that stability as conceived by Lyapunov has a limiting sense of an infinitesimal stability with infinitely small perturbations when the numbers ε_j, ε_j' tend to zero.

The Equations of Perturbed Motions

4. The solution of the question of stability depends on investigation of differential equations for the differences:

$$x_s = Q_s - F_s.$$

We shall term these equations the equations of perturbed motion and their obvious solution $x_1 = 0, \ldots, x_n = 0$, the unperturbed motion.

If the question of stability is studied with regard to the independent variables of a mechanical system, the differential equations of perturbed motion give the form of variation of the deviation or change of these variables.

Let us consider as an example a certain holonomic mechanical system under the action of conservative forces. Let $q_1, \ldots, q_k$ be its co-ordinates, $p_1, \ldots, p_k$ its momenta,

and $H(t, q_1, \ldots, q_k, p_1, \ldots, p_k)$, the Hamiltonian function. The equations of motion may be taken in the familiar canonical form:

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \qquad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}.$$

Let us assume that we intend to study the stability of motion with respect to the variables q_j, p_j, corresponding to some particular solution of the canonical equations:

$$q_j = q_j(t), \quad p_j = p_j(t).$$

Let the values of the co-ordinates q_j and the momenta p_j for the perturbed motion be:

$$q_j = q_j(t) + \xi_j, \quad p_j = p_j(t) + \eta_j,$$

where ξ_j, η_j are the deviations or variations of q_j, p_j, respectively. Denoting, for simplicity, the functions $q_j(t)$, $p_j(t)$ by q_j, p_j and noting that the perturbed motion is one of the motions of our system under the given forces, we have the following equations:

$$\frac{d(q_j + \xi_j)}{dt} = \frac{\partial H(t, q_i + \xi_i, p_i + \eta_i)}{\partial p_j},$$
$$\frac{d(p_j + \eta_j)}{dt} = -\frac{\partial H(t, q_i + \xi_i, p_i + \eta_i)}{\partial q_j},$$

where for conciseness we have put:

$$H(t, q_i + \xi_i, p_i + \eta_i) = \\ = H(t, q_1 + \xi_1, \ldots, q_k + \xi_k, p_1 + \eta_1, \ldots, p_k + \eta_k).$$

Expanding the right-hand side of these equations in Taylor series in the small deviations ξ_j, η_j and using the equations for unperturbed motion, we have:

$$\frac{d\xi_j}{dt} = \sum_i \left(\frac{\partial^2 H}{\partial p_j \partial q_i} \xi_i + \frac{\partial^2 H}{\partial p_j \partial p_i} \eta_i \right) + X_j,$$
$$\frac{d\eta_j}{dt} = -\sum_i \left(\frac{\partial^2 H}{\partial q_j \partial q_i} \xi_i + \frac{\partial^2 H}{\partial q_j \partial p_i} \eta_i \right) + Y_j,$$

where X_j, Y_j denote higher-order terms dependent on the deviations ξ, η. These are the equations of perturbed motion.

If we omit the terms X_j, Y_j from these equations, the

first approximation equations which remain have also been termed the variational equations of Poincaré.

When we consider the stability of the unperturbed motion of a mechanical system with respect to certain functions $Q_1, \ldots, Q_n$, dependent on t, q, p and holomorphic with respect to q, p, then from the explicit expressions for the differences:

$$x_s = Q_s(t, q_i+\xi_i, p_i+\eta_i) - Q_s(t, q_i, p_i) = \\ = \sum_i \left(\frac{\partial Q_s}{\partial q_i}\xi_i + \frac{\partial Q_s}{\partial p_i}\eta_i\right) + \cdots$$

there follows:

$$\frac{dx_s}{dt} = \sum_i \left(\frac{\partial Q'_s}{\partial q_i}\xi_i + \frac{\partial Q'_s}{\partial p_i}\eta_i\right) + \ldots,$$

where Q'_s denotes the total derivative with respect to time of the function Q_s, taken in accordance with the equations of motion

$$Q'_s = \frac{\partial Q_s}{\partial t} + \sum_j \left(\frac{\partial Q_s}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial Q_s}{\partial p_j}\frac{\partial H}{\partial q_j}\right).$$

If the functions Q_s are mutually independent and they are $n = 2k$ in number, then, expressing ξ_i, η_i in terms of x_s, in accordance with the previous relations, we may put the differential equations of perturbed motion in the normal form:

$$\frac{dx_s}{dt} = X_s,$$

where the functions X_s, which vanish when the variables x_s are all zero, will be holomorphic functions of the quantities $x_1, \ldots, x_n$, with coefficients which are known functions of time.

It may happen that it is possible to bring the equations of perturbed motion to the normal form if the number n of functions Q_s under investigation is less than twice the number of degrees of freedom of the mechanical system. We shall assume the number n and the functions Q_s to be such that the equations of perturbed motion may be brought to the given normal form.

5[2L]. In what follows we shall consider the differential equations of perturbed motion:

$$\frac{dx_s}{dt} = X_s \qquad (s = 1, \ldots, n), \tag{1}$$

leaving aside the initial equations of motion of the mechanical system, and the form of the functions Q_s, on the assumption that to every system of sufficiently small real perturbations ε_j, ε'_j, there corresponds some system of real initial values of the variables x_{s0} which, for an arbitrarily small positive number A, may always be made to satisfy the inequality:

$$\sum_s x_{s0}^2 < A,$$

the perturbations ε_j, ε'_j being subject to the condition that in absolute value they do not exceed sufficiently small but non-vanishing values of E_j, E'_j. We shall also assume that for arbitrarily small given positive numbers E_j, E'_j, it is always possible to find a positive value of A such that, to the magnitude $x_{10}^2 + \ldots + x_n^2$, not exceeding A, there correspond one or several systems of real values ε_j, ε'_j, the absolute magnitudes of which $|\varepsilon_j|$, $|\varepsilon'_j|$ are correspondingly smaller than E_j, E'_j.

Under this condition the initial values of the variables x_s may play the same role in solving the stability question as the quantities ε_j, ε'_j, if for given x_{s0} the variables x_s, satisfying equations (1), are completely defined. This last condition will be assumed to be always satisfied; therefore below, in place of the quantities ε_j, ε'_j, we shall always consider x_{s0}, transferring the term perturbation to the latter quantities.

We shall assume that the right-hand sides of the differential equations of perturbed motion (1), for t exceeding t_0, are expanded in convergent power series in integral powers of the variables $x_1, \ldots, x_n$, where the latter satisfy the condition:

$$\sum_s x_s^2 < A,$$

with the coefficients p_{sr}, $P_s^{(m_1 \ldots m_n)}$, which are real continuous functions of t in the expansion:

$$X_s = p_{s1}x_1 + \ldots + p_{sn}x_n + \sum P_s^{(m_1 \ldots m_n)} x_1^{m_1} \ldots x_n^{m_n},$$

The summation extends over all non-negative integers $m_1, \ldots, m_n$ satisfying the condition:

$$m_1 + \ldots + m_n > 1.$$

Lyapunov's definition of stability may be re-stated in terms of the variables x in the following manner:-

If for each given arbitrarily small number A, a positive number λ may be chosen such that for all perturbations $x_{10}, \ldots, x_{n0}$, satisfying the condition

$$\sum_s x_{s0}^2 \leqslant \lambda,$$

and for all values of t exceeding t_0, the inequality

$$\sum_s x_s^2 < A,$$

is satisfied, then the unperturbed motion is stable; in the opposite case it is unstable.

In the definition of stability it is assumed that there are no perturbation forces, i.e., that the perturbed motion occurs under the action of the same external forces which are taken into account in determining the unperturbed motion, while the number A is arbitrary and may be as small as required.

If the unperturbed motion is stable, the conditions required in the definition of stability will be satisfied, if only for some small but finite A and λ.

Lyapunov was not interested in the question of how great the value of λ might be, although in the proof of his theorem on stability he gave a useful practical method of finding λ for a given number A, bounded from above.

The stability problem in the presence of perturbing forces is meaningless, if the latter are in no way constrained. If, from case to case, the perturbing forces vary so little that their variation has no effect on the linear terms in the function X_s, the important practical problem arises of stability in the first approximation independently of the terms of higher order in the functions X_s. This problem was solved by Lyapunov, and its solution was his main achievement.

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CHAPTER 2

GENERAL THEOREMS OF LYAPUNOV'S DIRECT METHOD

Definitions

6. The direct method of studying stability developed by Lyapunov does not consist in integration of the differential equations of perturbed motion but in finding certain functions of the variables $t, x_1, \ldots, x_n$, the total derivatives of which (with respect to time) have certain properties by reason of equations (1).

As Lyapunov himself recognised, he was led to his method by studying the important memorandum of Poincaré, "On Curves Defined By Differential Equations"[1].

7[15L]. We shall consider real functions of the real variables $t, x_1, \ldots, x_n$ subject to the conditions

$$t \geqslant t_0 \quad \text{and} \quad \sum_s x_s^2 \leqslant H, \tag{2}$$

where t_0 and H are constants, where H is always assumed to be non-vanishing. We shall everywhere assume that these functions are continuous, single-valued and vanish when the variables x_s are all zero.

When, with the conditions (2) (if t_0 is made sufficiently large and H sufficiently small), the considered function V takes on, apart from zero, values of only one sign, such a function will be termed sign-constant. When we need to consider the sign, we shall refer to it as positive or negative.

If the sign-constant function V is independent of t while the constant H in (2) may be taken sufficiently small for the function V to vanish only when all the variables x_s vanish, then such a function V will be termed sign-definite, and when we need to draw attention to this sign, positive-definite or negative-definite.

The function V, depending explicitly on t, will be termed sign-definite only when it is possible to find a positive-definite function W independent of t such that one of the two expressions

$$V - W \quad \text{or} \quad -V - W$$

represents a positive function.

Thus, each of the functions

$$x_1^2 + x_2^2 - 2x_1x_2\cos t, \quad t(x_1^2 + x_2^2) - 2x_1x_2\cos t$$

is positive. But the first is only sign-constant while the second will be sign-definite if in the problem there are no other variables than x_1 and x_2; for the function W we may take $x_1^2 + x_2^2$, since for $t_0 > 2$ and arbitrary positive H the difference $V - W$ under condition (2) will never be negative.

In the space of variables $x_1, \ldots, x_n$ in the region bounded by the second of inequalities (2), the equation $V = c$ represents certain continuous surfaces in motion if V depends explicitly on t; the surface $V = 0$ passes through the origin, at which all variables x_s are assumed equal to zero.

If the function V is sign-definite and does not depend on t, the surface $V = c$ has the property that there is no continuous path from the origin to an arbitrary point of the sphere (A), where

$$\sum_s x_s^2 = A \qquad (A \leqslant H),$$

which does not intersect the surface $V = c$, the numerical value c being less than or equal to the smallest value of $|V|$ on this sphere. The locus of the first points of intersection of the surface $V = c$ with all possible continuous paths from the origin $(x_1 = 0, \ldots, x_n = 0)$ to points of the sphere (A) constitutes a certain closed surface which we shall term the $(n-1)$-dimensional cycle $V = c$. With continuous variation of c to zero the cycles $V = c$ represent closed surfaces nested one inside the other and contracting on to the origin for sufficiently small numerical values of the variables x_s.

If the function V is positive-definite and depends explicitly on t, then by definition there exists a positive-definite function W independent of t such that $V - W$ is a positive

function. If c does not at the same time exceed the smallest value of the function W on the sphere (A), then an arbitrary continuous path from the origin to an arbitrary point of the cycle $W=c$ must either intersect the surface $V=c$ at least once, or terminate on it. Denoting by "the cycle $V=c$" the locus of the first points of intersection of the surface $V=c$, for some $t>t_0$, with continuous paths from the origin to points of the cycle $W=c$, we note that for this the cycle $V=c$ is included within the cycle $W=c$.

If under conditions (2), $|V|$ does not exceed a certain finite number, the function V will be termed bounded. For a sufficiently small value of A, by continuity, all functions of V independent of t will be bounded. If the bounded function V is such that for all positive l chosen arbitrarily small there can be found a non-vanishing number λ such that for

$$t \geqslant t_0 \quad \text{and} \quad \sum_s x_s^2 \leqslant \lambda$$

the condition

$$|V| < l,$$

will be satisfied, we shall say that V has an infinitely small upper bound. This requirement is satisfied, by continuity, by each function V independent of t. However, functions dependent on t, no matter how bounded, may not satisfy it. For example, of the functions

$$\sin^2[(x_1^2+\ldots+x_n^2)t], \quad (x_1^2+\ldots+x_n^2)\sin^2 t$$

only the second has an infinitely small upper bound; neither of these positive functions is sign-definite.

If the function V has an infinitely small upper bound, no point of the surface $|V|=l$ in the space of variables x, no matter how small l is, ever enters the region

$$\sum_s x_s^2 \leqslant \lambda,$$

for $t \geqslant t_0$, where λ is some positive constant dependent on l.

The sign-definite condition for a function V gives a known limitation for the variation of the cycle $V=c$ outwards with increase of time; the cycle $W=c$ constitutes the outer

boundary of the region $W \leqslant c$ of the space $(x_1, \ldots, x_n)$, within which lies the cycle $V = c$ at an arbitrary moment of time $t \geqslant t_0$.

The condition of existence of an infinitely small upper bound to the function V gives a limitation on the variation of the surface $|V| = l$ inwards, in the sense that for each arbitrarily small positive l there exists a positive number λ different from zero defining the region

$$\sum_s x_s^2 \leqslant \lambda,$$

inside of which for each $t \geqslant t_0$ there are no points belonging to the surface $|V| = l$.

Simultaneously with the function V we shall consider its total derivative with respect to t, taken under the assumption that the variables x_s satisfy the differential equation of perturbed motion

$$V' = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x_1} X_1 + \ldots + \frac{\partial V}{\partial x_n} X_n.$$

Lyapunov's Stability Theorem

8[16L]. Theorem. If the differential equations of perturbed motion are such that it is possible to find a sign-definite function V, the derivative V' of which (by reason of these equations) would be a sign-definite function of opposite sign to V or identically equal to zero, the unperturbed motion is stable.

Proof. Let V be a positive-definite function and V' negative or zero. From the definition of a sign-positive function there can be found a positive-definite function W independent of t such that, for sufficiently large t_0 and sufficiently small H in the region defined by conditions (2), the inequalities

$$V' \leqslant 0, \quad V \geqslant W. \tag{3}$$

will be valid. It is necessary to show that for an arbitrary positive number A there can be found such a positive number λ that for initial perturbations x_{s0}, satisfying the inequality

$$\sum_s x_{s0}^2 \leqslant \lambda,$$

the values of the variables x_s varying in accordance with

the equations of the perturbed motion at the time $t=t_0$ with initial values x_{s0} for any time $t \geqslant t_0$, will satisfy the inequality

$$\sum_s x_s^2 < A.$$

Let A be a certain non-zero, arbitrarily small positive number (which in each case is assumed smaller than H); let l be the exact lower bound of the function W on the sphere (A)

$$\sum_s x_s^2 = A.$$

The number l necessarily differs from zero and is positive, since W is a positive definite function. Let us consider the function $V(t_0, x_1, \ldots, x_n)$; since this function does not depend explicitly on t, it has an infinitely small upper bound; consequently, for given l there is found a number λ such that, for values of the variables x_s satisfying the condition

$$\sum_s x_s^2 \leqslant \lambda,$$

the values of the functions $V(t_0, x_1, \ldots, x_n)$ will satisfy the inequality

$$V(t_0, x_1, \ldots, x_n) < l.$$

If the initial values x_{s0} of the variables x_s are chosen in accordance with the inequality

$$\sum_s x_{s0}^2 \leqslant \lambda,$$

then from the relationship

$$V - V_0 = \int_{t_0}^{t} V' \, dt,$$

where

$$V_0 = V(t_0, x_{10}, \ldots, x_{n0}) < l,$$

we find, in accordance with (3), that in their variation the variables x_s will, by reason of the differential equations of perturbed motion, satisfy the conditions

$$W \leqslant V \leqslant V_0 < l,$$

and thus the condition

$$\sum_s x_s^2 < A,$$

since l is the exact lower bound of the function W on the sphere (A).

The theorem is thus proved.

The function V satisfying the conditions of this theorem will be termed the Lyapunov function*.

In the proof of his theorem, given as an ε-proof, there should be mentioned the method, useful in practice, proposed by Lyapunov for finding for a given positive number A less than H, with the aid of functions V and W, the positive number λ, having the following property: if the initial values of x_{s0} are subject to the inequality

$$\sum_s x_{s0}^2 \leqslant \lambda,$$

then for all subsequent time the values of the variables x_s will satisfy the inequality

$$\sum_s x_s^2 < A.$$

Lyapunov uses only the necessary fact that for arbitrary positive A, no matter how small, the corresponding λ exists; he did not consider the question of the largest value of λ for a given A.

Example. Let us consider the equations of perturbed motion in the form

$$\frac{dx}{dt} = -(x - \beta y)(1 - ax^2 - by^2),$$

$$\frac{dy}{dt} = -(y + \alpha x)(1 - ax^2 - by^2)$$

with positive constants α, β, a, b; and let $a < b$, $\alpha < \beta$.

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* The problem on the existence of the Lyapunov function for each stable unperturbed motion was solved by Professor Persidskii (Dokl. Akad. Nauk SSSR, 1938).

The positive-definite function

$$V = \alpha x^2 + \beta y^2$$

satisfies the conditions of the theorem; its total derivative with respect to time in accordance with the given equations is negative for sufficiently small values of x, y:

$$V' = -2(\alpha x^2 + \beta y^2)(1 - ax^2 - by^2).$$

The unperturbed motion $x = 0, y = 0$ is therefore stable.

Let us illustrate the proof. The region $x^2 + y^2 \leqslant H$, where the functions V are positive-definite and the derivative V has negative and zero values, is defined by the relationship

$$H = \frac{1}{b}.$$

Let us consider a certain circle $x^2 + y^2 = A$; the ellipse

$$\alpha x^2 + \beta y^2 = A\alpha.$$

is tangent to it from within. The square of the radius of the maximum circle inscribed in this ellipse is

$$\lambda = \frac{A\alpha}{\beta}.$$

If $A < H$, the motion beginning from an arbitrary point x_0, y_0, lying inside, this circle

$$x_0^2 + y_0^2 < \lambda.$$

does not bring the moving point x, y outside the circle $x^2 + y^2 = A$, since in the region under consideration the value of the derivative V' is negative and, consequently, the point $P(x, y)$ moving in the field of similar and similarly located ellipses $V = \text{const}$ in accordance with the given equations should pass to the inner ellipses. This also proves that starting from the given initial position (x_0, y_0), a moving point never reaches the ellipse

$$\alpha x^2 + \beta y^2 = A\alpha,$$

lying inside the circle

$$x^2 + y^2 \leqslant A.$$

Since the quantity λ so defined can be found for each arbitrarily small value of A, we conclude that the unperturbed motion is stable.

9[16L]. Corollary. If for the differential equations of perturbed motion there exist integrals $U_1, \ldots, U_m$, which vanish when all the variables x_s are simultaneously zero, and if there be found a function V satisfying conditions (3), for the same significance of W as before, only for the variables for which $U_1 = 0, \ldots, U_m = 0$, we would conclude that the unperturbed motion is stable for perturbations subject to the equations $U_1 = 0, \ldots, U_m = 0$.

A particular case of this corollary is a theorem of Routh[2]. Essentially, let $q_1, \ldots, q_k$ be independent Lagrangian co-ordinates of a certain integral mechanical system for which T is the kinetic energy and U the potential function. Let us assume that the variables $q_{s+1}, \ldots, q_k$ $(s < k)$ are cyclical in the sense that

$$\frac{\partial L}{\partial q_\alpha} = 0 \quad (\alpha = s+1, \ldots, k),$$

where $L = T+U$ denotes the so-called Lagrangian function. The equations of motion of such a system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i'}\right) - \frac{\partial L}{\partial q_i} = 0 \qquad (i = 1, \ldots, k)$$

have $k-s$ first integrals

$$\frac{\partial L}{\partial q_\alpha'} = \beta_\alpha,$$

where β_α are constants.

If the function R be introduced, defined by the equality

$$R = L - \sum_\alpha q_\alpha' \beta_\alpha,$$

and in it the q_α' be eliminated using the above first integrals, we obtain an expression for R as a function of

$$t,\ q_1, \ldots, q_s,\ q_1', \ldots, q_s',\ \beta_{s+1}, \ldots \beta_k.$$

The variation of this last is, if $j = 1, \ldots, s;\ \alpha = s+1, \ldots, k,$

$$\delta R = \sum \frac{\partial R}{\partial q_j}\delta q_j + \sum_1 \frac{\partial R}{\partial q_j'}\delta q_j' + \sum_\alpha \frac{\partial R}{\partial \beta_\alpha}\delta\beta_\alpha;$$

now the variation of this function R, in accordance with the formula defining it, is

$$\delta R = \sum_j \frac{\partial L}{\partial q_j}\delta q_j + \sum_j \frac{\partial L}{\partial q_j'}\delta q_j' - \sum_\alpha q_\alpha'\delta\beta_\alpha.$$

A comparison of coefficients for identical variations in these two expressions leads to the relationships

$$\frac{\partial L}{\partial q_j} = \frac{\partial R}{\partial q_j}, \quad \frac{\partial L}{\partial q_j'} = \frac{\partial R}{\partial q_j'}, \quad -q_\alpha' = \frac{\partial R}{\partial \beta_\alpha},$$

in accordance with which Lagrange's equations for non-cyclical co-ordinates take the form

$$\frac{d}{dt}\left(\frac{\partial R}{\partial q_j'}\right) - \frac{\partial R}{\partial q_j} = 0 \qquad (j = 1, \ldots, s).$$

The function R is independent of the cyclic co-ordinates q_α and their velocities q_α'. In a way the last equations ignore the cyclic co-ordinates and reduce the dynamic problem to that of the motion of a mechanical system with the new Lagrangian function R, and with a smaller number of degrees of freedom. The values of the cyclic co-ordinates q_α after integration of the last equations are defined by the quadratures

$$q_\alpha = -\int \frac{\partial R}{\partial \beta_\alpha}\,dt.$$

Let us suppose that for certain values of β_α the equations of motion permit a particular solution $q_j = 0$ $(j = 1, \ldots, s)$.

To this solution there corresponds a stationary motion in which only the cyclic co-ordinates q_α will vary. The equations of perturbed motion (with fixed values of the constants β_α) will be the equations for non-cyclical co-ordinates q_j with the Lagrangian function R.

If R is not explicitly dependent on t these equations have a first integral corresponding to the integral of kinetic energy

$$H \equiv \sum_j \frac{\partial R}{\partial q_j'} q_j' - R = h.$$

Routh's theorem states that if the function $V = H - H_0$ is sign-definite in the variables q_j, q_j', then (in accordance with Lyapunov's theorem, since $V' \equiv 0$) the stationary motion will be stable provided that the values of the constants β_α are not perturbed; H_0 denotes the function H when in the latter all non-cyclical co-ordinates q_j and their velocities q_j' are set equal to zero. It is interesting to note that the stationary motion may be stable when the function $H - H_0$ is not sign-definite.

10. Example. Let us consider a rigid body with one fixed point O. Let the moving co-ordinate axes $Oxyz$ coincide with the principal axes of the ellipsoid of inertia of the body, found for the fixed point. The moments of inertia of the body with regard to the axes x, y, z are denoted by A, B, C respectively.

Lagrange's case is obtained when $A = B$ and the co-ordinates of the centre of gravity of the body are $x = 0$, $y = 0$, $z > 0$.

Let p, q, r be the projections of the instantaneous angular velocity on the principal axes, Ox, Oy, Oz of the ellipsoid of inertia; γ, γ', γ'' the directional cosines of the vertical z_1 with respect to the moving axes x, y, z.

Practical interest is centred in the problem of stability of vertical rotation:

$$p = 0, \quad q = 0, \quad r = r_0, \quad \gamma = 0, \quad \gamma' = 0, \quad \gamma'' = 1.$$

Many investigations have been concerned with this problem. We shall study the question of stability with regard to the variables

$$p, \ q, \ r, \ \gamma, \ \gamma', \ \gamma''.$$

Let us denote the variation of the variables for perturbed motion as follows:

$$p = \xi, \quad q = \eta, \quad r = r_0 + \zeta, \quad \gamma = \alpha, \quad \gamma' = \beta, \quad \gamma'' = 1 + \delta.$$

In order to solve this stability problem, we shall seek a Lyapunov function among integrals of the equation of perturbed motion.

In Lagrange's example the following integrals are well-known:

$$A(p^2+q^2)+Cr^2+2mgz\gamma''=h,$$
$$A(p\gamma+q\gamma')+Cr\gamma''=k,$$
$$\gamma^2+\gamma'^2+\gamma''^2=1,$$
$$r=r_0.$$

From this the equations of perturbed motion have these integrals:

$$V_1=A(\xi^2+\eta^2)+C(\zeta^2+2r_0\zeta)+2mgz\delta,$$
$$V_2=A(\xi\alpha+\eta\beta)+C(\delta\zeta+r_0\delta+\zeta),$$
$$V_3=\alpha^2+\beta^2+\delta^2+2\delta,$$
$$V_4=\zeta.$$

Lyapunov's function will be sought in the form of the following sum of quadratic forms

$$V=V_1+2\lambda V_2-(mgz+Cr_0\lambda)V_3+\mu V_4^2-2(Cr_0+C\lambda)V_4=$$
$$=A\xi^2+2\lambda A\xi\alpha-(mgz+Cr_0\lambda)\alpha^2+$$
$$+A\eta^2+2\lambda A\eta\beta-(mgz+Cr_0\lambda)\beta^2+$$
$$+(C+\mu)\zeta^2+2\lambda C\delta\zeta-(mgz+Cr_0\lambda)\delta^2.$$

In order for the first two quadratic forms of the same type to be positive, it is necessary and sufficient to select λ such that their discriminant is positive, i.e.,

$$\begin{vmatrix} A & A\lambda \\ A\lambda & -(mgz+Cr_0\lambda) \end{vmatrix} > 0$$

or

$$A\lambda^2+Cr_0\lambda+mgz<0.$$

The last inequality is possible if the polynomial has two distinct real roots, i.e.,

$$\begin{vmatrix} A & \frac{Cr_0}{2} \\ \frac{Cr_0}{2} & mgz \end{vmatrix} < 0$$

or

$$C^2r_0^2-4Amgz>0.$$

The third form will be positive if its discriminant is positive, i.e.,

$$\begin{vmatrix} C+\mu & C\lambda \\ C\lambda & -(mgz+Cr_0\lambda) \end{vmatrix} > 0$$

or

$$\left(\frac{C^2}{C+\mu}\right)\lambda^2+Cr_0\lambda+mgz<0.$$

To reduce this problem to that already considered, we take μ such that

$$\frac{C^2}{C+\mu}=A,$$

or

$$\mu=\frac{C(C-A)}{A}.$$

Here

$$C+\mu=\frac{C^2}{A}>0.$$

Thus if

$$C^2r_0^2-4Amgz>0,$$

then λ may be taken so that the quadratic integral V is positive-definite with respect to all variables $p, q, r, \alpha, \beta, \delta$. This, in accordance with Lyapunov's stability theorem, leads to the conclusion as to the stability of the vertical rotation of Lagrange's spinning top.

11[16L]. Complement to Lyapunov's theorem on asymptotic stability. If the sign-definite function V has an infinitesimal upper bound while its derivative V' is a sign-definite function of opposite sign, then each perturbed motion sufficiently close to the unperturbed motion will approach the latter asymptotically.

Proof. If the functions V and V' are sign-positive then, by definition, there exists t_0 and a positive number H such that for $t \geqslant t_0$ and for values of the variables x_s satisfying the condition

$$\sum_s x_s^2 \leqslant H,$$

there exist positive-definite functions W, W' independent of time, such that if V is positive,

$$V - W \geqslant 0 \quad \text{and} \quad -V' - W' \geqslant 0. \tag{4}$$

Let us determine the region of initial perturbations

$$\sum x_{s0}^2 \leqslant \lambda,$$

for which the values of the variables x_s do not leave the region,

$$\sum x_s^2 < A,$$

as in the proof of the fundamental stability theorem of Lyapunov (section 8). Let l be the exact lower bound of the function W on the sphere (A). Then for λ we select a number defining the region

$$\sum x_s^2 \leqslant \lambda,$$

into which enters not one point of the surface

$$V(t_0, x_1, \ldots, x_n) = l.$$

It is easy to verify that for arbitrary perturbations x_{s0}, lying in the region (λ)

$$\sum_{s} x_{s0}^2 \leqslant \lambda,$$

with the above properties of the functions V and V', it is not possible to find such a positive number e which would be smaller than all values taken on by the function V in the resulting perturbed motion for $t \geqslant t_0$. In fact, if for certain initial values x_{s0} there existed such a number e, then by the properties of the function V as a function possessing an infinitesimal upper bound, there would exist such a number ε defining the region (ε)

$$\sum_{s} x_s^2 \leqslant \varepsilon,$$

for arbitrary points of which the values of V would be less than e, Consequently, if such e exists, the values of the variables x_s would lie somewhere in the region

$$\varepsilon \leqslant \sum_s x_{s0}^2 \leqslant A.$$

Let us denote by l' the exact lower bound of the function W' in this closed region. This lower bound must be some positive number, since W' is a positive definite function. Hence, in accordance with the second of conditions (4), for arbitrary $t \geqslant t_0$ the values of the function in this region satisfy the condition

$$-V' \geqslant W' \geqslant l' > 0;$$

from the equation

$$V - V_0 = \int V' dt$$

we derive

$$V \leqslant V_0 - l''(t - t_0).$$

But this is impossible as the left-hand side of the inequality is a positive-definite function of t, and the right-hand side becomes negative for sufficiently large t.

Thus, for arbitrarily small e, there is always a time when the function V becomes smaller than e. Being a decreasing function of t, V will then always remain smaller than e. Consequently, if for e we take the exact lower bound of the function W in the region

$$\mu \leqslant \sum_s x_s^2 \leqslant A,$$

there is always a time when the function V becomes and remain smaller than e; at least from this moment onwards, the values of the variables x_s will always remain in the region

$$\sum_s x_s^2 < \mu.$$

From this we note that for all initial perturbations x_{s0} lying in the region (λ), the values of the variables x_s tend to zero as t increases indefinitely.

12. If the differential equations of perturbed motion are such that there exists a function $V(t, x_1, \ldots, x_n)$ such that the function

$$V - \theta(t) W \qquad (\theta(t_0) = 1)$$

is "positive-constant" for positive-definite and time-independent W and for monotonic increase to infinity of the function $\theta(t)$, with increase in t, while its total derivative with respect to time V' is "negative-constant" or zero, then the unperturbed motion is asymptotically stable while the region of possible values of the variables $x_1, \ldots, x_n$ is defined by the inequality

$$W \leqslant \frac{V_0}{\theta(t)},$$

in which V_0 denotes the value of the function V at the initial moment t_0 for the initial values of the variables $x_{10}, \ldots, x_{n0}$.

The simplest definitions of the region of possible values of $x_1, \ldots, x_n$ are obtained if the function W is taken as $\mu(x_1^2 + \ldots + x_n^2)$, where μ is a positive constant.

Instability Theorem

13. To show that the unperturbed motion is unstable, it is sufficient to find even a single trajectory emerging from the given region (2) for arbitrarily small numerical values of perturbations x_{s0}.

The set of values of the variables x_s, satisfying the inequality $V > 0$ under the assumptions (2), will be termed the region $V > 0$, and the surface $V = 0$ the boundary of the latter. If the function V depends explicitly on t, then with variation of t the region $V > 0$ will also vary.

If the function W is bounded in the region $V > 0$ in such a manner that for all arbitrarily small positive l there can be found a non-zero number λ such that for

$$t \geqslant t_0, \quad \sum_s x_s^2 \leqslant \lambda, \quad V \geqslant 0$$

the inequality

$$|W| < l,$$

will be satisfied, we say that W has an infinitely small upper bound in the region $V > 0$. This requirement is

satisfied by every function W having an infinitesimal upper bound.

The function $W(t, x_1, \ldots, x_n)$ will be termed sign-definite in the region $V > 0$ if it can vanish in this region only at the boundary $V = 0$ and if for arbitrarily small positive ε there is found such a non-zero number l that for x_s satisfying the condition $|V| \geqslant \varepsilon$, and for each $t > t_0$ the inequality

$$|W| \geqslant l.$$

is valid.

Obviously the function

$$\lambda V + W,$$

will be sign-definite in the region $V > 0$ if W is a positive or identically vanishing function while λ is a non-zero positive constant. If V is independent of t, each function W independent of t will be sign-definite in the region $V > 0$ if it vanishes nowhere within the region $V > 0$ while it may vanish on the boundary of the region (i.e. for $V = 0$). Each function V is positive-definite in its region $V > 0$.

Each sign-definite function U will be sign-definite in the region $V > 0$, if the function V has an infinitely small upper bound in the region $V > 0$. Actually, if V has an infinitely small upper bound in the region $V > 0$ then, by definition, for given arbitrarily small ε, there can be found a number λ such that the region $V > \varepsilon$ will lie outside the sphere

$$\sum x_s^2 = \lambda.$$

If U is a sign-definite function, say positive, then by definition there will exist a positive-definite function W independent of t such that the function

$$U - W$$

will be non-negative. Therefore an exact lower bound (necessarily positive) of the function W in the region

$$\lambda \leqslant \sum x_s^2 \leqslant H$$

will be the lower bound for the values of the function U in the region $V > \varepsilon$.

Theorem. If the differential equations of perturbed motion are such that it is possible to find a function V bounded in the region $V > 0$, existing for all $t \geqslant t_0$ and for arbitrarily small absolute values of the variables x_s, with derivative V' positive-definite in the region $V > 0$ by reason of these equations, the unperturbed motion is unstable.

Proof. For a bounded function V there are found constants t_0 and H such that for all values of the variables x_s in the region $V > 0$, satisfying also conditions (2), the inequality

$$V < L. \tag{5}$$

will be satisfied.

It is necessary to prove that for such H there does not exist a sufficiently small positive λ such that for arbitrary initial perturbations x_{s0}, subject to the equation

$$\sum x_{s0}^2 = \lambda,$$

the second of the inequalities (2) is not violated for some $t > t_0$.

We shall prove this by reductio ad absurdum. Let us assume that such a number λ exists. We shall take the initial perturbations x_{s0} on the sphere (λ) such that the initial value of the function V_0 is non-zero and positive. For arbitrarily small V_0 and for V' positive-definite in the region $V > 0$, there is found a non-zero number l', such that for the variables x_s satisfying the conditions

$$V \geqslant V_0,$$

the values of the function V' will satisfy the inequality $V' \geqslant l'$.

Therefore as long as the inequality $V \geqslant V_0$ is not violated for all values of t greater than t_0, from the equation

$$V - V_0 = \int_{t_0}^{t} V' \, dt$$

we derive the inequality

$$V \geqslant V_0 + l'(t - t_0).$$

This inequality may exist simultaneously with inequality (5) only for values of t less than

$$t_0+\frac{L-V_0}{l'}.$$

Violation of inequality (5) means in this case that the second of conditions (2) has been violated. And thus the unperturbed motion is shown to be unstable.

14[16L]. Lyapunov proposed two theorems on instability which are obtained very simply from the foregoing.

First theorem. If the differential equations of perturbed motion are such that it is possible to find a function V which by reason of these equations would have a sign-definite derivative V', would admit an infinitely small upper bound and would be such that, for each value of t greater than a certain constant, it would be possible to make the quantity V have the same sign as its derivative by an appropriate choice of magnitudes x_s (arbitrarily small in absolute value), then the unperturbed motion is unstable.

For proof it is sufficient to note that the function V of this proposition satisfies the conditions formulated in the theorem of section 13, as the sign-definite derivative V', say positive, will be positive-definite in the region $V>0$ since V admits an infinitely small upper bound.

Second theorem. If the differential equations of perturbed motion are such that it is possible to find a bounded function V the derivative of which by reason of these equations could be put in the form

$$\frac{dV}{dt}=\lambda V+W,$$

where λ is a positive constant and W either identically zero or representing a certain sign-constant function, and if, in the latter case, there is found a function V such that for each t greater than a certain number, it is possible by suitable choice of magnitudes x_s, numerically arbitrarily small, to make it a magnitude of the same sign as W, then the unperturbed motion is unstable.

In fact, the derivative V is a sign-definite function in the region where V has a sign agreeing with the sign of W if W is not identically zero. If W is identically equal to

zero, then V' will be sign-definite both in the region $V >$ and in the region $V < 0$.

In all cases when this is possible, we shall employ Lyapunov's theorem* to establish instability.

15[16L]. Example. Let the equation of perturbed motion be

$$\frac{dx_s}{dt} = \frac{\partial V}{\partial x_s} \qquad (s = 1, \ldots, n),$$

where V is a holomorphic function of the variables x_s independent of t, the expansion of which begins with terms of not lower than second order.

We have

$$V' = \left(\frac{\partial V}{\partial x_1}\right)^2 + \ldots + \left(\frac{\partial V}{\partial x_n}\right)^2.$$

Therefore whenever V is a negative-definite function the unperturbed motion will be stable. On the other hand, the unperturbed motion will be unstable if V may take on positive values in an arbitrarily small region around the point $x_1 = 0, \ldots, x_n = 0$, and so long as we are not concerned with a case where the system of equations

$$\frac{\partial V}{\partial x_s} = 0$$

may be satisfied by real values of the variables x_s arbitrarily small in absolute magnitude but not simultaneously equal to zero. This case, of course, does not occur if the determinant

$$\left\| \frac{\partial^2 V}{\partial x_s \partial x_r} \right\|$$

is non-zero when all variables x_s are zero.

Example. Let us consider the system of differential equations

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n,$$

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* It is possible to note yet a further theorem on instability where two functions are introduced[3].

where the continuous bounded functions p_{sr} (t) have the property $p_{sr} = -p_{rs}$ for $r = s$.

If the coefficients p_{ss} are either zero or may vanish for certain values of $t \geqslant t_0$, remaining negative for other values, or if for $t \geqslant t_0$ they are all negative, then the stability of unperturbed motion follows from the fact that positive-definite function V

$$V = \frac{1}{2} \sum x_s^2$$

has, in accordance with the given equations, a negative or identically vanishing total derivative with respect to t given by

$$V' = \sum_{s,\ k} p_{sk} x_s x_k = \sum_s p_{ss} x_s^2.$$

If for arbitrary $t \geqslant t_0$, the inequalities $p_{ss} < -h < 0$ are valid, where h is a positive constant, the derivative V' will be negative-definite; in this case the unperturbed motion is stable and a neighbouring perturbed motion will tend asymptotically to it.

If p_{ss} satisfies the conditions $p_{ss} > k > 0$, where k is a certain positive number, then in this case the unperturbed motion is unstable.

Example. Stability of constant rotation. Euler's case, of the motion of a rigid body with a single fixed point, will arise when the centre of gravity of the body coincides with the fixed point. Let us study the stability of the particular solution

$$p_0 = 0, \quad q_0 = 0, \quad r = r_0 > 0.$$

The variation of the variables in the perturbed motion will be denoted by:

$$p = \xi, \quad q = \eta, \quad r = r_0 + \zeta.$$

The equations of perturbed motion will be:

$$A \frac{d\xi}{dt} = (B - C) \eta (r_0 + \zeta),$$

$$B \frac{d\eta}{dt} = (C - A) (r_0 + \zeta) \xi,$$

$$C \frac{d\zeta}{dt} = (A - B) \xi \eta.$$

The question of the stability of constant rotation about the largest $(A \geqslant B > C)$ and the smallest $(A \leqslant B < C)$ semi-axis of the ellipsoid of inertia is determined from the existence of a sign-definite integral of the equation of perturbed motion

$$\frac{A-C}{B}\xi^2+\frac{B-C}{A}\eta^2=\pm[A\xi^2+B\eta^2+2Cr_0\zeta+C\zeta^2]^2.$$

The instability of rotation about the central axis of the ellipsoid of inertia $(A < C < B)$ can be shown by examination of the function

$$V=\xi\eta.$$

Its derivative is given by

$$V'=(r_0+\zeta)\left[\frac{B-C}{A}\eta^2+\frac{C-A}{B}\xi^2\right].$$

If $\zeta+r_0$ vanishes the instability is obvious. If $\zeta+r_0>0$, then V' will be positive-definite in the region $V>0$. The function V, not dependent explicitly on t, has an infinitely small upper limit; on the basis of the instability theorem (section 13) we deduce that the rotation about the central semi-axis is unstable.

Example. The region $V>0$ may consist of several individual zones. For application of the theorem of section 13 only one connected zone C of the region $V>0$ can be of interest. To determine C by a single inequality $W>0$ it is sufficient to consider the continuous function W, equal to V in the zone C and equal to $-|V|$ outside of C.

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CHAPTER 3

STABILITY OF EQUILIBRIUM IN THE PRESENCE OF CONSERVATIVE FORCES

Lagrange's Theorem

In his formulations Torricelli established a theorem on the stability of the equilibrium positions of mechanical systems, which was generalised by Lagrange for arbitrary conservative forces. For the most elementary cases, Lyapunov gave a converse to Lagrange's theorem. In considering the case where the potential function depends on a certain parameter, Poincaré established the beginnings of the theory of equilibrium branching.

16. Consider a mechanical system constrained by certain holonomic time-independent couplings, in equilibrium under the action of conservative forces.

Let us denote by $q_1, \ldots, q_k$ its independent Lagrangian co-ordinates. With no loss in generality, it is always possible to assume that for the considered equilibrium position of the dynamical system the values of all variables q_s are equal to zero. Let T denote the kinetic energy and U the potential function of the forces acting on the system. With couplings independent of time the kinetic energy T represents a positive definite quadratic form of the velocities $q_1', \ldots, q_k'$. The potential function U is assumed to depend only on the co-ordinates $q_1, \ldots, q_k$ (which are taken to be zero for the equilibrium position).

Under these assumptions the differential equations of perturbed motion will be the same as the equations of motion of the mechanical system considered, if questions of stability are investigated with regard to the co-ordinates $q_1, \ldots, q_k$ and the velocities $q_1', \ldots, q_k'$.

Lagrange's Theorem. If in the equilibrium position the potential function U has an isolated maximum, then this equilibrium position is stable.

Proof. If the potential function U has in the equilibrium position an isolated maximum and is equal to zero, then at least for sufficiently small values of the variables q_s (small in absolute magnitude) the values of the function U will be negative. This means that, for small values of the variables q_s, the function U is a negative-definite function of the Lagrangian co-ordinates $q_1, \ldots, q_k$. Therefore the total energy of the mechanical system

$$H = T - U$$

will constitute a positive-definite function with respect to the co-ordinates $q_1, \ldots, q_k$ and the velocities $q_1', \ldots, q_k'$. Its total derivative with regard to time vanishes

$$H' \equiv 0,$$

since for the equations of motion with couplings independent of time the integral of energy exists. Consequently, H satisfies all the conditions of Lyapunov's theorem on stability, which proves Lagrange's theorem.

17. Let us now consider the case where in the region of values of the variables q_j (small in absolute magnitude) there exists a region where the potential function U (vanishing when all q_s are zero) has a positive value. The potential function U should define the forces acting on the dynamical system, therefore it should possess everywhere in the region considered (let us say, continuous) partial derivatives.

The differential equations of perturbed motion will be taken, for simplicity, in the form of the canonical Hamiltonian equations

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j},$$

where the momenta p_j are defined by the formulae

$$p_j = \frac{\partial T}{\partial q_j'}.$$

These equations have the integral of energy $H = T + U = h$.

Let us consider only the case where the potential function U is some homogeneous function of degree m with regard to the variables q_s

$$U = U_m$$

and where, for arbitrarily small absolute magnitudes of the variables q_s, it may take on a positive value.

The equilibrium position is here unstable[1, 2].

Let us consider the function

$$W = -H \sum_j p_j q_j.$$

In the region of small absolute magnitudes of the coordinates $q_1, \ldots, q_k$ and momenta $p_1, \ldots, p_k$ we determine the region C existing under our assumptions for arbitrarily small absolute magnitudes of the variables q_j, p_j, defined by the simultaneous inequalities:

$$H < 0 \text{ and } \sum_j p_j q_j > 0.$$

The total derivative with regard to t of the function W, by reason of the differential equations of perturbed motion, has the form

$$W' = -H\left[\sum_j p_j \frac{dq_j}{dt} + \sum_j q_j \frac{dp_j}{dt}\right].$$

Substituting here the explicit expressions

$$H = T - U_m \text{ and } p_j = \frac{\partial T}{\partial q_j'},$$

in accordance with the equations of motion and Euler's theorem on homogeneous functions, we have:

$$W' = -H\left[2T + mU_m - \sum_j \frac{\partial T}{\partial q_j} q_j\right].$$

In the region C considered, the value of the total energy H should be negative by definition of the region C; but since $H = T - U_m$, in this region C the following relation should apply:

$$U_m > 0.$$

Since the couplings applied to the system are assumed independent of the time t, the kinetic energy of the system T

constitutes a positive-definite function with regard to the momenta p_j

$$T = \sum_{s,r} g_{sr} p_s p_r$$

in which the coefficients g_{sr} are certain functions of the co-ordinates $q_1, \ldots, q_k$. For all possible values of the co-ordinates q_j all principal diagonal minors of the kinetic energy discriminant

$$\|g_{rs}\|$$

should be positive; in the equilibrium position, where all co-ordinates q_s are equal to zero, these principal minors are equal to certain positive quantities bounded from below. Therefore the principal diagonal minors of the discriminant of the form

$$\sum_{s,r} \left(2g_{sr} - \sum_j \frac{\partial g_{sr}}{\partial q_j} q_j\right) p_s p_r$$

(quadratic with regard to p_j) will also be positive for sufficiently small absolute magnitudes of the co-ordinates q_s and the form

$$2T - \sum \frac{\partial T}{\partial q_j} q_j$$

will be positive-definite with regard to p_s. Thus, the expression in square brackets in the equation for W' will be positive in the region C; in other words, in the region C, for numerically sufficiently small values of the co-ordinates q, the values of the derivative W' will be positive, i.e. of the same sign as W.

In the region C the function W satisfies all conditions of the instability theorem of section 13, which proves the instability of the considered position of the system. Just as simply, starting from the previous definition of the function W in the region C it is possible to prove the instability of the equilibrium position when the potential function has the form

$$U = U_m + U_{m+1} + \cdots,$$

where for arbitrarily small q_s the function U may take on positive values and in the region C the sign of expressions

$$U_m + U_{m+1} + \ldots \text{ and } mU_m + (m+1)U_{m+1} + \ldots$$

is defined by the form U_m.

The case when $m = 2$ and when the sign of the potential function U is defined by second-order terms, without the necessity of considering terms of higher-orders, was first studied by Lyapunov[3].

Poincaré's Stability Coefficients

18. For concrete applications of Lagrange's theorem it is useful to know the criterion for the minimum of the potential function $V = -U$.

In a region close to the equilibrium position of a dynamical system where the variables q_s all have zero values, the expansion of the function V in a Maclaurin series begins in the general case with the quadratic terms

$$V = \frac{1}{2} \sum_{i,j} a_{ij} q_i q_j + \cdots,$$

since the free term $V(0, \ldots, 0)$ vanishes in accordance with condition of section 16 and the linear terms of the expansion vanish by reason of the equilibrium equations

$$\left(\frac{\partial V}{\partial q_s}\right)_0 = 0.$$

Summation extends over all possible values of the indices $i, j = 1, \ldots, k$; for conciseness of notation the second partial derivatives of V in the equilibrium position are denoted by a_{ij},

$$a_{ij} = \left(\frac{\partial^2 V}{\partial q_i \, \partial q_j}\right)_0 = a_{ji}.$$

By linear transformation of the variables q_s the quadratic form

$$2f = \sum_{i,j} a_{ij} q_i q_j$$

may be reduced to a sum of squares. In reality, in the k-dimensional auxiliary Euclidian space we construct the mutually orthogonal axes $q_1, \ldots, q_k$ and consider the equation of the quadric

$$2f = \sum a_{ij} q_i q_j = 1.$$

Its left-hand side will be in the form of a sum of squares if

the co-ordinate axes coincide with the principal axes of the quadric. A quadric axis, by definition, is a straight line passing through the centre and the apex or, otherwise, through that point of the quadric where the radius vector coincides with the normal.

The condition of collinearity of the normal to the quadric and the radius vector has the form

$$\frac{\partial f}{\partial q_i} = \varkappa q_i \qquad (i = 1, \ldots, k),$$

where $\varkappa$ is a hitherto undefined factor; or in explicit form

$$\sum_j a_{ij} q_j = \varkappa q_i \qquad (i = 1, \ldots, k). \tag{6}$$

This system will have a non-vanishing solution if the determinant composed of the coefficients is equal to zero:

$$\Delta(\varkappa) = \begin{vmatrix} a_{11} - \varkappa & \ldots & a_{1k} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{k1} & \ldots & a_{kk} - \varkappa \end{vmatrix} = 0.$$

This is called the secular equation; it defines k real roots $\varkappa_1, \ldots, \varkappa_k$, closely related to the magnitudes of the semi-axes of the quadric.

Sylvester's Theorem. The roots of the secular equation are all real.

Proof. If $\varkappa$ is a root of a secular equation the system (6) has for q_s a non-vanishing solution. If this root $\varkappa$ is complex, then q_s will also be complex. Let $\bar{q}_s$ denote the complex quantity conjugate to q_s.

Let us multiply the equations of system (6) respectively by $\bar{q}_i$ and add

$$\sum_{ij} a_{ij} \bar{q}_i q_j = \varkappa \sum_i q_i \bar{q}_i;$$

the conjugate expression will be

$$\sum_{ij} a_{ij} q_i \bar{q}_j = \bar{\varkappa} \sum_i \bar{q}_i q_i.$$

From this, if the symmetry of coefficients $a_{ij} = a_{ji}$ of the

quadratic form f under consideration is taken into account, we immediately obtained the relationship

$$\varkappa = \frac{\sum a_{ij}\bar{q}_i q_j}{\sum \bar{q}_i q_i} = \frac{\sum a_{ji}\bar{q}_i q_j}{\sum \bar{q}_i q_i} = \bar{\varkappa},$$

which proves that the real roots $\varkappa$ are real, since only a real quantity is its own conjugate.

To determine the geometric significance of the roots of the secular equation $\varkappa$, let us multiply equations (6) by q_i and add

$$2f = \varkappa \sum_i q_i^2.$$

If the root $\varkappa$ is positive then (since the solutions q_s of the system (6) satisfying the equation of the quadric $2f = 1$, constitute the co-ordinates of the apex) the expression $\sum q_i^2$ represents the square of the corresponding semi-axis of the quadric. If the root $\varkappa$ is negative, the co-ordinates of the apex of the corresponding semi-axis, by virtue of the last relationship, will be purely imaginary while $\sum q_i^2$ will represent the square of the imaginary semi-axis of the quadric $2f = 1$.

In both cases in accordance with the last relationship the numerical value of $\varkappa$ is equal to the reciprocal of the square of the corresponding semi-axis of the quadric.

Let $\varkappa_1, \ldots, \varkappa_k$ be the roots of the secular equation and $a_1, \ldots, a_k$ the corresponding semi-axes of the quadric, real for positive roots and imaginary for negative. From the well-known canonical form of the quadric equation expressed in terms of its axes

$$\sum_i \frac{x_i^2}{a_i^2} = 1$$

it follows that it is possible to bring a given form f, by means of linear transformations to new variables x_i, to the form

$$2f = \sum_i \varkappa_i x_i^2.$$

The roots of the secular equation $\varkappa_1, \ldots, \varkappa_k$ define the character of the form f. If they are all positive then the

form f, and together with it the potential energy $V=f+\ldots$, will be positive-definite and have a minimum in the equilibrium position. If among the roots $\varkappa$ there is one which is negative, the potential energy $V=f+\ldots$ for the corresponding choice of numerically arbitrarily small values of the variables q_s may be made negative. Poincaré called the roots $\varkappa$ the stability coefficients and the number of negative $\varkappa_s$, the degree of instability.

Using these terms, the theorems of Lagrange and Lyapunov may be expressed as follows: if in the equilibrium position all stability coefficients are positive, the equilibrium is stable; if there is at least one negative stability coefficient, the equilibrium is unstable.

19. Example. A homogeneous tri-axial ellipsoid lies on a horizontal plane. Determine the Poincaré stability coefficients for its equilibrium position.

Let the equation of the ellipsoid with respect to its axes be given by:

$$Ax^2+By^2+Cz^2-1=0.$$

The equilibrium positions of the tri-axial ellipsoid will be those positions in which the normal to the ellipsoid at the point of tangency will be collinear with the radius from the centre at the point of tangency; in other words, the equilibrium positions will be positions in which the point of tangency with the fixed plane coincides with the apex of one of the semi-axes.

The distance from the centre (of gravity) to the tangent plane

$$AxX+ByY+CzZ-1=0$$

at the point of tangency (x, y, z) is

$$\delta=\frac{1}{\sqrt{A^2x^2+B^2y^2+C^2z^2}}.$$

From this, we see that, if this tangent plane is the horizontal plane on which the ellipsoid lies, the potential function in the region of the equilibrium position $(x=0,\ y=0)$ has the form

$$V=-U=mg\delta=\frac{mg}{\sqrt{A(A-C)x^2+B(B-C)y^2+C}}=$$
$$=\frac{mg}{\sqrt{C}}\left[1-\frac{1}{2}\left(\frac{A(A-C)}{C}x^2+\frac{B(B-C)}{C}y^2\right)+\dots\right],$$

and consequently the stability coefficients of the equilibrium position $x=0$, $y=0$ are

$$\varkappa_1=-\frac{mg}{2\sqrt{C}}\frac{A(A-C)}{C},\quad \varkappa_2=-\frac{mg}{2\sqrt{C}}\frac{B(B-C)}{C}.$$

Thus if the ellipsoid lies on the apex of the minor semi-axis ($A<B<C$) this equilibrium position is stable ($\varkappa_1>0, \varkappa_2>0$); if the ellipsoid lies on the apex of the central semi-axis ($A<C<B$) then this equilibrium is unstable with a single degree of instability; if the ellipsoid lies on the apex of the major semi-axis ($A>B>C$), the equilibrium is unstable with two degrees of instability.

Criteria For The Quadratic Forms To Be Sign-Definite

20. From the point of view of stability, the magnitudes of Poincaré's stability coefficients are not so important as their signs. To determine the sign of the root it is not necessary to solve the secular equation; it is sufficient to know the necessary and sufficient conditions for the potential function to be positive-definite.

To solve this last problem it is possible to use the familiar Hurwitz criterion for the real parts of the roots of an algebraic polynomial to be negative, in the present case the polynomial being $\Delta(-\lambda)$. The simplest method, and the most convenient for calculation, is Lagrange's method, consisting in the successive extraction of complete squares from the quadratic form. Kronecker's method for reducing a quadratic form to a sum of squares is interesting since it leads to inequalities which are very simply related to Sylvester's criterion. The real quadratic form

$$\varphi=\sum_{ij}c_{ij}x_ix_j\quad (c_{ij}=c_{ji})$$

is positive-definite when and only when all principal diagonal minors of its discriminant are positive:

$$c_{11}>0,\ \begin{vmatrix}c_{11}&c_{12}\\c_{21}&c_{22}\end{vmatrix}>0,\ \dots,\ \begin{vmatrix}c_{11}&\dots&c_{1n}\\ \dots&\dots&\dots\\c_{n1}&\dots&c_{nn}\end{vmatrix}>0.$$

Proof. For concise notation let us denote the principal diagonal minors of the discriminant by

$$C_r = \begin{vmatrix} c_{11} \cdots c_{1r} \\ \cdot \quad \cdot \quad \cdot \\ c_{r1} \cdots c_{rr} \end{vmatrix},$$

and the minor (inclusive of sign) of the discriminant C_n, corresponding to the element c_{rs} by C_{rs}.

By transformation of the variables

$$x_i = u_i + \frac{C_{in}}{C_{nn}} u_n \quad (i < n),$$

$$x_n = u_n$$

the form φ is brought to the form

$$\varphi = \sum_{i,j=1}^{n-1} c_{ij} u_i u_j + \frac{C_n}{C_{n-1}} u_n^2,$$

which may be verified by direct substitution. From this we see that the ratio $\frac{C_n}{C_{n-1}}$ is equal to one of the stability coefficients of the form φ in the variables $u_1, \ldots, u_n$. Continuing this process further, on the form

$$\sum_{1}^{n-1} c_{ij} u_i u_j,$$

we verify the sufficiency of the conditions of the theorem.

The necessity of these conditions follows from the fact that the above transformation is possible for arbitrary signed C_r. If $C_n = 0$, the quadratic form may be brought to a sum of less than n squares, and consequently the form will not be positive-definite. Thus, the condition $C_n \neq 0$ is necessary. If $C_n \neq 0$, then among its first-order minors is found at least one which is non-vanishing; with a suitable numbering of the variables x_j this may be considered to be C_{n-1}. Continuing this treatment further, we conclude that the variables x_j may be distributed so that C_r be different from zero for arbitrary r. Then the form φ, reduced by these transformations to squares, will have as coefficients of the squares the ratios $\frac{C_r}{C_{r-1}}$.

Therefore φ may be a positive-definite form only under the conditions of the theorem since the condition $C_1 = c_{11} > 0$ is obvious.

Note. We shall sometimes consider a more general quadratic form in the complex variables x_j with complex coefficients

$$\varphi = \sum_{ij} c_{ij} \bar{x}_i x_j,$$

for which $c_{ij} = \overline{c_{ji}}$. These forms are termed Hermitian. The condition of sign-definiteness and particularly the positiveness of the form φ have the same form as with the Sylvester criterion*.

21. Example. A dynamical system with three degrees of freedom exists under the influence of forces, the potential function of which has the form

$$V = \frac{1}{2}[a^2(q_1^2 + q_2^2) + 2aq_3(q_1 + q_2) + b^2 q_3^2],$$

where q_s are independent co-ordinates of the system, and a, b, α are certain real constants. Let the position for which all q_s vanish be an isolated equilibrium position. It is required to determine the stability condition for this equilibrium position.

In accordance with Lagrange's theorem this equilibrium position will be stable if the potential function V has a minimum there.

For V to be positive-definite its discriminant

$$\begin{vmatrix} a^2 & 0 & \alpha \\ 0 & a^2 & \alpha \\ \alpha & \alpha & b^2 \end{vmatrix}$$

should have all principal diagonal minors positive.

From this there follow the conditions

$$a^2 > 0, \quad a^2 b^2 - 2\alpha^2 > 0.$$

Example. A free particle of mass m moves in space under the action of forces, the potential function of which (in cylindrical co-ordinates r, θ, z) has the form $V = \varphi(r, z)$.

The kinetic energy

* A proof of this hypothesis may be found by the reader in[4] where can also be found a more complete proof of Sylvester's criterion (in section 20).

$$T=\frac{m}{2}(r'^2+r^2\psi'^2+z'^2)$$

and the potential function do not depend explicitly on ψ; therefore ψ is a cyclical co-ordinate (section 9), to which corresponds the integral $mr^2\psi'=\beta$. From this, Routh's function has the form

$$R=\frac{m}{2}(r'^2+z'^2)-\varphi-\frac{\beta^2}{2mr^2}.$$

For steady-state motion the conditions

$$\frac{\partial R}{\partial r}=0,\quad \frac{\partial R}{\partial z}=0,$$

should be satisfied; let these conditions be satisfied by $r=r_0$, $z=z_0$. This steady-state motion will be stable in accordance with Routh's theorem if the modified potential function

$$W=\varphi+\frac{\beta^2}{2mr^2}$$

is a minimum. Let

$$r=r_0+\xi,\quad z=z_0+\eta;$$

then

$$W(r,\ z)-W(r_0,\ z_0)=$$
$$=\frac{1}{2}\left[\varphi_{rr}\xi^2+2\varphi_{rz}\xi\eta+\varphi_{zz}\eta^2+\frac{3}{r_0}\varphi_r\xi^2\right]+\ldots,$$

where φ_r, φ_{rr} are the values of the partial derivatives of φ with respect to these variables for the steady-state motion under consideration. Therefore, for stability all principal diagonal minors of the discriminant

$$\begin{vmatrix}\varphi_{rr}+\frac{3}{r_0}\varphi_r & \varphi_{rz}\\ \varphi_{rz} & \varphi_{zz}\end{vmatrix}$$

should be positive. This gives two inequalities:

$$\varphi_{rr}+\frac{3}{r_0}\varphi_r>0,\quad \left(\varphi_{rr}+\frac{3}{r_0}\varphi_r\right)\varphi_{zz}-\varphi_{rz}^2>0.$$

Example. Let the potential function V of some dynamical system depend on the co-ordinates x, y, which define the position of the system and let V have a minimum in the equilibrium position x_0, y_0.

Let us introduce the notation

$$X=\left(\frac{\partial V}{\partial x}\right)_y, \quad Y=\left(\frac{\partial V}{\partial y}\right)_x,$$

where the indices indicate the quantities which remain constant in taking the partial derivatives. The stability conditions for this equilibrium position are, by reason of Sylvester's criterion,

$$\left(\frac{\partial X}{\partial x}\right)_y>0, \quad \left(\frac{\partial X}{\partial x}\right)_y\left(\frac{\partial Y}{\partial y}\right)_x-\left(\frac{\partial X}{\partial y}\right)_x^2>0, \quad \left(\frac{\partial Y}{\partial y}\right)_x>0.$$

Let us consider that the equilibrium condition $X=0$ is violated through some variation of the quantity x; if the quantity y remains constant, the change in X is defined by the derivative

$$\left(\frac{\partial X}{\partial x}\right)_y.$$

If the magnitude y changes, restoring the second equation of equilibrium $Y=0$, the change in X will be defined by the derivative

$$\left(\frac{\partial X}{\partial x}\right)_{Y=0}.$$

$$\left(\frac{\partial X}{\partial x}\right)_{Y=0}=\frac{\partial(X, Y)}{\partial(x, Y)}=\frac{\dfrac{\partial(X, Y)}{\partial(x, y)}}{\dfrac{\partial(x, Y)}{\partial(x, y)}}=\left(\frac{\partial X}{\partial x}\right)_y-\frac{\left(\dfrac{\partial X}{\partial y}\right)_x^2}{\left(\dfrac{\partial Y}{\partial y}\right)_x}.$$

Consequently, if

$$\left(\frac{\partial X}{\partial y}\right)_x$$

does not vanish, then

$$0<\left(\frac{\partial X}{\partial x}\right)_{Y=0}<\left(\frac{\partial X}{\partial x}\right)_y.$$

These inequalities express certain particular cases of the LeChatellier-Braun principle.

Bifurcation of Equilibrium

22. In many problems of mechanics the potential energy of a material system depends not only on the variables defining the position of the system but also on certain parameters. For example, in the problem of a

compressed rod the potential function V depends not only on the variables defining the form of the rod in the perturbed state but on the longitudinal load P.

The general theory of equilibrium positions of such systems with various values of the parameters was established by Poincare.

Let the potential energy V of some dynamical system depend on the co-ordinates $x_1, \ldots, x_n$, defining the position of the system, and on a single variable parameter α.

For fixed value of the parameter α the equilibrium positions are defined by the equations

$$\frac{\partial V}{\partial x_s} = 0 \qquad (s = 1, \ldots, n).$$

Let

$$x_1^{(i)} = \varphi_1^{(i)}(\alpha), \ldots, x_n^{(i)} = \varphi_n^{(i)}(\alpha)$$

represent a linear sequence of real roots of the equilibrium equations; we assume that the functions $\varphi_j^{(i)}(\alpha)$ are continuous functions of the parameter α.

In the $(n+1)$-dimensional space of the variables $(\alpha, x_1, \ldots, x_n)$ the last equations define the lines C_i, forming in conjunction some real curve B, different points of which correspond to differing possible states of equilibrium.

Individual branches C_i intersect at points where at least two real roots of the equilibrium equation coincide. In the neighbourhood of such points the equations

$$\frac{\partial V}{\partial x_s} = 0 \; (s = 1, \ldots, n)$$

do not have a single-valued solution; at such points the functional determinant[5]

$$\Delta = \frac{\partial\left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right)}{\partial (x_1, \ldots, x_n)} = \left\| \frac{\partial^2 V}{\partial x_i \, \partial x_j} \right\|.$$

should vanish. In other words, if at a certain point M on curve B the Hessian of potential energy Δ vanishes, such a point may be a point of intersection of the branches C_i; the points M are termed critical or points of bifurcation (branching).

Certain points of the curve B correspond to stable states of equilibrium and certain to unstable. The changes in stability on branches C_i may occur only at points of bifurcation. In fact, for a particular branch C_i the coordinates x_s are certain functions of the parameter α, $x_s = \varphi_s^{(i)}(\alpha)$ and, consequently, for the branch C_i the determinant is also a function of α, if in Δ we substitute $x_s = \varphi_s^{(i)}(\alpha)$. In passing along the branch C_i from point to point the value of Δ will change.

For a definite state of equilibrium corresponding to a certain point of the branch C_i the stability is defined by Poincaré coefficients or, otherwise, by the roots of the corresponding secular equation

$$\Delta(\varkappa) = \left\| \frac{\partial^2 V}{\partial x_i \partial x_j} - \delta_{ij}\varkappa \right\| = 0.$$

Here, and everywhere below, δ_{ij} is Kronecker's symbol; δ_{ij} is equal to zero if i and j are different and is equal to unity if i and j are equal. The expansion of the polynomial $\Delta(\varkappa)$ in linear factors $\Delta(\varkappa) = (\varkappa_1 - \varkappa) \ldots (\varkappa_n - \varkappa)$ gives directly

$$\Delta = \Delta(0) = \varkappa_1 \ldots \varkappa_n.$$

Since at the point of bifurcation $\Delta = 0$, at least one of the Poincaré stability coefficients $\varkappa_i$ vanishes there, and vice versa.

But the roots of the secular equation $\Delta(\varkappa) = 0$ are always real. Therefore on the branch C_i the stability of a linear sequence of equilibria may vanish (or appear) only at points of bifurcation, since with loss (or gain) of stability at least one of the stability coefficients will pass through zero and change its sign.

Let M be some point of bifurcation; the value of the parameter α corresponding to it will be taken equal to zero.

In various problems of mechanics the real branches of the equilibrium curve are of primary interest. Let us note without proof two propositions relating to this [6].

Consider that in some way the real branches $C_1, \ldots, C_k$ of the curve B passing through the bifurcation point M have been found. Let us plot two planes $\alpha - \varepsilon = 0$ and $\alpha + \varepsilon = 0$ where ε is assumed to be a sufficiently small positive

constant such that each of the branches $C_1, \ldots, C_k$ intersects the plane $\alpha^2 - \varepsilon^2 = 0$.

Let the known branches $C_1, \ldots, C_k$ intersect the plane $\alpha - \varepsilon = 0$ at points P_i and the plane $\alpha + \varepsilon = 0$ at the points Q_i. It is possible to determine the signs of the Hessian Δ for the points P_i and Q_i.

It is found that in addition to the branches $C_1, \ldots, C_k$ there pass through the point M at least a further r real branches of the equilibrium curve B

$$r = \frac{1}{2}\left|\sum [\Delta_{P_i}] - \sum [\Delta_{Q_i}]\right|,$$

where the summation extends over all these points while $[f]$ denotes $+1$ if $f > 0$ and -1 if $f < 0$.

The distribution of stable and unstable equilibrium positions for fixed values of α is subject to a law of stability shift. The stability equations for fixed values of the parameter $\alpha = \delta$ are defined in the plane $\alpha - \delta = 0$ of the space $(\alpha, x_1, \ldots, x_n)$, corresponding to equilibrium states of the point P. The equations

$$\alpha = \delta, \quad \frac{\partial V}{\partial x_2} = 0, \ldots, \frac{\partial V}{\partial x_n} = 0$$

define a certain line L_1, lying in the plane $\alpha - \delta = 0$ and passing through all points of equilibrium. Taking some direction of circling the connected parts of the line L_1 (without dual points) we may number the points P_i in the order of encountering them. Let P_i and P_{i+1} belong to some connected portion of the line L_1. It is found that

$$[\Delta_{P_i}] + [\Delta_{P_{i+1}}] = 0,$$

and since $\Delta = x_1 \ldots x_n$, we see that in passing from equilibrium P_i to equilibrium P_{i+1} the degree of instability changes by an odd number.

23. These general theorems on the number of real branches passing through a point of bifurcation, and on the change of stability, have a very trivial sense for that case of practical interest when in the region of the bifurcation point the determinant

$$\Delta_1 = \frac{\partial\left(\frac{\partial V}{\partial x_2}, \ldots, \frac{\partial V}{\partial x_n}\right)}{\partial(x_2, \ldots, x_n)}.$$

is non-zero. In this case, close to the bifurcation point it is possible to provide a unique solution for the system of equations with regard to the variables $x_2, \ldots, x_n$

$$\frac{\partial V}{\partial x_2} = 0, \ldots, \frac{\partial V}{\partial x_n} = 0,$$

to obtain

$$x_i = x_i(x_1, \alpha) \qquad (i = 2, \ldots, n)$$

and, employing these solutions, to eliminate the variables $x_2, \ldots, x_n$. After this substitution we shall have only a single equilibrium equation

$$\frac{\partial V}{\partial x_1} = f(x_1, \alpha) = 0.$$

The auxiliary space thus formed will then be the plane (α, x_1). On the equilibrium curve B defined in this plane by the equation $f = 0$, bifurcation is defined by points where

$$\delta \equiv \frac{\partial f}{\partial x_1} = 0.$$

vanishes.

Let the line $\alpha = \text{const}$ intersect the equilibrium line $f = 0$ at points P_j which we number successively in encircling the line $\alpha = \text{const}$ from left to right. The expression δ obviously changes sign in passing from the point P_i to the point P_{i+1}, if both of these points are points of bifurcation. The significance of δ is not difficult to determine. When the mechanical system has only a single degree of freedom, δ is the single Poincaré stability coefficient. In the general case we have:

$$\delta = \frac{\partial^2 V}{\partial x_1^2} + \sum_{s=2}^{n} \frac{\partial^2 V}{\partial x_1 \partial x_s} \frac{\partial x_s}{\partial x_1}.$$

If the solution $x_i = x_i(x_1, \alpha)$ is substituted in the initial equations, we obtain an identity which may be differentiated with regard to the variable x_1:

$$0 = \frac{\partial^2 V}{\partial x_1 \partial x_i} + \sum_{s=2}^{n} \frac{\partial^2 V}{\partial x_i \partial x_s} \frac{\partial x_s}{\partial x_1} \qquad (i = 2, \ldots, n).$$

The preceding relationship and this system of equations

will have a non-trivial solution for

$$1, \frac{\partial x_2}{\partial x_1}, \ldots, \frac{\partial x_n}{\partial x_1},$$

when the determinant consisting of their coefficients vanishes, i.e.,

$$\begin{vmatrix} -\delta + \frac{\partial^2 V}{\partial x_1^2} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_n} \\ \cdots & \cdots & \cdots \\ \frac{\partial^2 V}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_n^2} \end{vmatrix} = 0,$$

from which we obtain

$$\delta = \frac{\Delta}{\Delta_1}.$$

In accordance with the reduction of the quadratic forms to the canonical form by Kronecker's method we note that δ is equal to one of the stability coefficients in the corresponding variables. From the assumed relationship $\Delta_1 \neq 0$ it follows that to each bifurcation point $\Delta = 0$ in the complete system of variables $(\alpha, x_1, \ldots, x_n)$ there corresponds a bifurcation point $\delta = 0$ of the reduced system, and vice versa. At a point of bifurcation one and only one stability coefficient δ will vanish. In fact, if two stability coefficients vanish, the canonical form of the quadratic form in the expansion of V should reduce to the sum of $n-2$ squares and therefore all first-order minors of the determinant Δ should vanish, which contradicts the assumption $\Delta_1 \neq 0$.

Let us consider the interesting case when, in the reduction of the quadratic form V to its canonical form by Kronecker's method, all the coefficients of the reduced form except δ are always positive. Then the stability or instability of equilibrium will depend only on the sign of δ; with δ positive the equilibrium is stable, with negative δ it is unstable. In the plane (x_1, α) where the x_1-axis passes horizontally from left to right while the α-axis passes from below upwards, those branches of equilibrium $f = 0$ will be stable for which the values of f to the right are positive $(\delta > 0)$.

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CHAPTER 4

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

Particular Solutions

If in the equations of perturbed motion the right-hand sides are explicit functions of time, Routh has called the corresponding unperturbed motion transient; when they are not explicitly dependent on t, the unperturbed motion is termed steady-state. In the latter case the variational equations will have constant coefficients.

24. Let us consider a system of linear differential equations

$$\frac{dx_i}{dt} = p_{i1}x_1 + \ldots + p_{in}x_n \quad (i = 1, \ldots, n).$$

with constant coefficients p_{ij}.

We shall look for a particular solution of the for

$$x_i = \left(A_{1i}\frac{t^m}{m!} + A_{2i}\frac{t^{m-1}}{m-1!} + \ldots + A_{m+1,\,i}\right)e^{\lambda t},$$

where A_{kj} and λ are certain constants; among the constants A_{1i}, some should be non-vanishing; a positive integral power m as large as possible must be determined for each λ.

If this expression is substituted in the system of differential equations, after cancellation of the common factor $e^{\lambda t}$, we obtain:

$$\left(A_{1i}\frac{t^{m-1}}{m-1!} + \ldots + A_{mi}\right) + \\ + \lambda\left(A_{1i}\frac{t^m}{m!} + A_{2i}\frac{t^{m-1}}{m-1!} + \ldots + A_{m+1,\,i}\right) = \\ = \sum_j p_{ij}\left(A_{1j}\frac{t^m}{m!} + A_{2j}\frac{t^{m-1}}{m-1!} + \ldots + A_{m+1,\,j}\right).$$

These relationships should be valid for arbitrary values

of t; therefore coefficients with common powers of t are to be equated:

$$\begin{cases} \sum_j (p_{ij} - \delta_{ij}\lambda) A_{1j} = 0, \\ \sum_j (p_{ij} - \delta_{ij}\lambda) A_{2j} = A_{1i}, \\ \cdots\cdots\cdots\cdots\cdots \\ \sum_j (p_{ij} - \delta_{ij}\lambda) A_{m+1,j} = A_{mi} \\ (i = 1, \ldots, n). \end{cases} \tag{7}$$

The first of these systems of equations (7) will then have a non-trivial solution for A_{1j} when the determinant of its coefficients vanishes

$$\Delta(\lambda) = \begin{vmatrix} p_{11} - \lambda & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} - \lambda & \cdots & p_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} - \lambda \end{vmatrix} = 0.$$

The roots of this characteristic equation define the value of the characteristic λ.

Let $\lambda = \lambda_0$ be a certain root of the characteristic equation. The first of the systems (7) defines a single non-trival solution for the constants A_{1j} if at least one of the first-order minors of the determinant $\Delta(\lambda_0)$ is non-vanishing. It defines k linearly dependent solutions for A_{1j} if the root λ_0 causes all the minors of the determinant $\Delta(\lambda_0)$ up to the order $k-1$ inclusive to vanish, while one of the k-th-order minors at least does not vanish.

25. Let us consider first the case where the root $\lambda = \lambda_0$ does not cause one first-order minor (at least) to vanish. Without loss of generality we shall assume that the latter is Δ_{11}. Then the solution of the first of systems (7) may be written (if it be taken into account that we are insterested in the particular solution to within a factor common to all A_{1j}) by the well-known formulae

$$A_{1j} = \Delta_{1j} \quad (j = 1, \ldots, n), \tag{8}$$

where Δ_{ij} denotes the minor (inclusive of sign) of the determinant $\Delta(\lambda)$, corresponding to the element in the i-th row and j-th column; it is assumed that $\lambda = \lambda_0$. Among the constants A_{1j} under the above assumptions at least one is

non-zero, namely A_{11}.

If λ remains a variable parameter and does not take on the value of the root $\lambda = \lambda_0$ or any other fixed value, the relationships (8) define A_{1j} in the form of certain polynomials in λ. If these polynomials A_{1j} are substituted in the equations $i = 2, \ldots, n$ of the first of systems (7), we obtain the relationship

$$\sum_j (p_{ij} - \delta_{ij}\lambda) A_{1j} = 0 \qquad (i = 2, \ldots, n),$$

which will be satisfied identically with regard to λ, since the left-hand sides are determinants of n-th-order with two identical rows. These identities may be differentiated with regard to λ as many times as desired.

Differentiating them k times in succession $(k = 1, 2, \ldots)$, we conclude that the polynomials

$$A_{k+1,\,j} = \frac{1}{k!} A_{1j}^{(k)}, \tag{9}$$

where $A_{1j}^{(k)}$ signifies the k-th derivative of A_{1j} with regard to λ, satisfy the equations $(i = 2, \ldots, n)$ of any system of (7) independently of the value of λ.

Substitution of the polynomials A_{1j}, defined by formulae (8) for arbitrary λ, in the first equation of the first of the systems (7) gives the relationship

$$\sum_j (p_{1j} - \delta_{1j}\lambda) A_{1j} = \Delta(\lambda).$$

In accordance with this relationship the first system of (7) will satisfy the values of the polynomials A_{1j} for λ, equal to a root of the characteristic equation $\Delta(\lambda) = 0$.

Let $\lambda = \lambda_0$ be a simple root of the characteristic equation. Then the polynomials A_{kj}, defined by formulae (8) and (9), for values of λ coinciding with the root λ_0 cannot satisfy the second system of (7) and, consequently, the highest power m in the solution considered by us is equal to zero. In fact, if $\lambda = \lambda_0$ is a simple root of the determinant $\Delta(\lambda)$, the value of the first derivative of $\Delta(\lambda)$ with regard to λ should be non-zero for $\lambda = \lambda_0$. Therefore if the latter relationship is differentiated once and the definition of A_{2j} used according to formulae (9), we obtain for $\lambda = \lambda_0$

$$\Delta'(\lambda_0) = \sum_j (p_{1j} - \delta_{1j}\lambda_0) A_{2j} - A_{11} \neq 0;$$

thus the first equation of the second system of (7) is not satisfied. Let now $\lambda = \lambda_0$ be a repeated root, of multiplicity μ. Under this assumption the values of all derivatives of the determinant $\Delta(\lambda)$ up to the order $\mu - 1$ inclusive will vanish for $\lambda = \lambda_0$, while the μ-th-order derivative for $\lambda = \lambda_0$ will be non-vanishing. Differentiating the last relationship k times in succession ($k = 1, 2, \ldots$) and employing definition (9)

$$\frac{1}{k!}\Delta^{(k)}(\lambda) = \sum_j (p_{1j} - \delta_{1j}\lambda) A_{k+1,j} - A_{k1},$$

we conclude that the values of the polynomials A_{rj} ($r = 1, \ldots, \mu$) for $\lambda = \lambda_0$ will satisfy the first μ systems of (7) and not satisfy the $(\mu+1)$-st. Consequently, the highest power of t will be in this case one less than the multiplicity of the root, that is, $m = \mu - 1$.

The values found for the constants A_{kj} satisfy not only the complete system (7), but all of its parts obtained by elimination of a certain number of systems of equations from (7). This fact proves the existence of particular solutions of the form investigated, in which the degree of the secular terms begins with an arbitrary number less than m. In other words, simultaneously with the required particular solutions of the highest power m for the given root

$$x_i = f_i(t)\, e^{\lambda t},$$

where

$$f_i(t) = A_{1i}\frac{t^m}{m!} + \ldots + A_{m+1,i},$$

there exist its derivatives, particular solutions of the form

$$x_i = f_i^{(k)}(t)\, e^{\lambda t},$$

where $f_i^{(k)}(t)$ denotes the k-th derivative with regard to t of the polynomial $f_i(t)$ ($k = 1, \ldots, m$). Thus, for this characteristic root $\lambda = \lambda_0$ of multiplicity $\mu = m + 1$ we determine the complete group of μ particular solutions, obviously linearly independent of each other since the highest powers of t in the secular terms of these equations are all distinct.

26. Let us now consider the general case where the root $\lambda = \lambda_0$ causes all minors of the determinant $\Delta(\lambda)$ up to the order $k - 1$ inclusive to vanish, but causing at least one

of the k-th-order minors not to vanish.

In this case the first of systems (7) defines k linearly independent solutions for A_{1j}. These k linearly independent solutions will be determined in the following manner. Let $(\lambda - \lambda_0)^{\mu_r}$ be the factor of the highest degree μ_r in the highest common factor D_{n-r} of the r-th-order minors of the determinant $\Delta(\lambda)(r = 1, \ldots, k)$. Considering derivatives of these minors with regard to λ and noting that they are expressed linearly in terms of the minors of order higher by unity, we verify the existence of the inequalities

$$\mu > \mu_1 > \cdots > \mu_{k-1} > \mu_k = 0,$$

where μ signifies the multiplicity of the root $\lambda = \lambda_0$ for the characteristic determinant $\Delta(\lambda)$.

Among the r-th-order minors we find the minor (inclusive of sign)*

$$\Delta_{i_1 \ldots i_r, j_1 \ldots j_r},$$

obtained after dropping from the determinant Δ the rows $i_1, \ldots, i_r$ and the columns $j_1, \ldots, j_r$, which would have a root $\lambda = \lambda_0$ of multiplicity exactly μ_r, while the minor $\Delta_{i_1 \ldots i_{r-1}, j_1 \ldots j_{r-1}}$ has a root $\lambda = \lambda_0$ of multiplicity μ_{r-1}. Assuming λ to be different from λ_0, we form the expressions

$$A_{1, j_s} = \frac{\Delta_{i_1 \ldots i_r, j_1 \ldots j_{r-1} j_s}}{(\lambda - \lambda_0)^{\mu_r}} \qquad (s = r, r+1, \ldots, n). \tag{10}$$

They define $A_{1j_s} (s = r, \ldots, n)$ in the form of polynomials in λ, of which at least one, namely A_{1j_r}, does not vanish for $\lambda = \lambda_0$. The $A_{1j_1}, \ldots, A_{1j_{r-1}}$ not entering into this definition will be equated to zero. The expressions $A_{kj}, (k = 2, 3, \ldots)$ are defined here in accordance with formula (9).

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* The minor

$$\Delta_{i_1 \ldots i_r, j_1 \ldots j_r}$$

obtained by eliminating from the determinant the rows with indices $i_1, \ldots, i_r$ and the columns with indices $j_1, \ldots, j_r$, will be given the sign $(-1)^{i_1 + \ldots + i_r + j_1 + \ldots + j_r + \sigma}$, where σ denotes the sum of the indices of disturbed order in the rows $i_1, \ldots, i_r$ and $j_1, \ldots, j_r$.

The polynomials A_{1j} thus defined are substituted in the first of systems (7)

$$\sum_j (p_{ij} - \delta_{ij}\lambda) A_{1j} = \\ = \frac{1}{(\lambda - \lambda_0)^{\mu_r}} \sum_{j_s} (p_{ij_s} - \delta_{ij_s}\lambda) \Delta_{i_1 \dots i_r, j_1 \dots j_{r-1} j_s}.$$

The sum on the right-hand side will vanish identically, as soon as $i = i_k$ when $k > r$, since then it will be proportional to a determinant in which two rows will be identical. If $i = i_k$ while $k \leqslant r$, then, without defining the sign, we have:

$$\sum_j (p_{i_k i} - \delta_{i_k i}\lambda) A_{1j} = \frac{\pm \Delta_{i_1 \dots i_{k-1} i_{k+1} \dots i_r, j_1 \dots j_{r-1}}}{(\lambda - \lambda_0)^{\mu_r}}.$$

In accordance with the choice of $\Delta_{i_1 \dots i_r, j_1 \dots j_r}$, on the right-hand side of this relationship is a polynomial in λ, having a root $\lambda = \lambda_0$ of multiplicity not less than $\mu_{r-1} - \mu_r$ if $k < r$ and exactly equal to $\mu_{r-1} - \mu_r$ if $k = r$.

Thus, the expressions A_{1j} adopted in the first of systems (7) will satisfy identically all equations $i = i_k$ independently of the values of λ if $k > r$, while the equations for $k = 1, \dots, r$ will be satisfied in accordance with the last relationship if we set $\lambda = \lambda_0$.

Differentiating with respect to λ the identically satisfied equations $(i = i_k,\ k > r)$ of the first of systems (7), we note that for our defined quantities A_{1j} all analogous equations $(i = i_k,\ k > r)$ will be satisfied in each of the systems (7) independently of the value of λ.

Since the right-hand sides of the latter relationships $(i = i_k,\ k \leqslant r)$ have $\lambda = \lambda_0$ as a root in each case of multiplicity not less than $\mu_{r-1} - \mu_r$, the values of all derivatives of these expressions up to the order $\mu_{r-1} - \mu_r - 1$ inclusive will vanish for $\lambda = \lambda_0$ while this, in accordance with (9), denotes that the remaining equations $(i = i_k,\ k \leqslant r)$ of the first, second, ..., $(\mu_{r-1} - \mu_r)$-th systems of (7) will be satisfied by the values of the polynomials A_{kj} for $\lambda = \lambda_0$. In the $(\mu_{r-1} - \mu_r + 1)$-st system there will be at least one equation not satisfied, its $i = i_r$-th equation, since the $(\mu_{r-1} - \mu_r)$-th derivative of

$$\frac{\Delta_{i_1 \dots i_{r-1}, j_1 \dots j_{r-1}}}{(\lambda - \lambda_0)^{\mu_r}}$$

is non-vanishing for $\lambda = \lambda_0$.

Thus, the values of the polynomials A_{kj} for $\lambda = \lambda_0$ taken from the definitions (10) and (9) satisfy the $\mu_{r-1} - \mu_r$ systems of (7), and, consequently, the highest degree m for such definitions will be $m = \mu_{r-1} - \mu_r - 1$.

Applying this procedure to the minors of order $r = 1, \ldots, k$, we obtain k particular solutions with highest powers $m = \mu_{r-1} - \mu_r - 1$. Each of these solutions, as has been remarked previously, gives a group of $\mu_{r-1} - \mu_r$ solutions by differentiation, with respect to t, of the polynomials occurring in the solution expressed as a product with factors $e^{\lambda t}$.

Thus, for the characteristic root $\lambda = \lambda_0$ of multiplicity μ, k groups of solutions ($r = 1, \ldots, k$) are found, each of which consists of $\mu_{r-1} - \mu_r$ solutions. The total number of solutions found is obviously equal to the multiplicity of the root

$$\sum_{r=1}^{k} (\mu_{r-1} - \mu_r) = \mu.$$

The solutions found are linearly independent. In fact, we have seen that the solutions of any individual group are linearly independent, since they have differing highest powers of t in their secular terms. Let the chain of initial minors

$$\Delta_{i_1 \ldots i_r, j_1 \ldots j_r}$$

be defined by some common indices for $r = 1, \ldots, k$. The variables x_{i_1}, entering into all solutions of the group, corresponding to the minor $r = 1$, will be zero by definition for all other groups of particular solutions. Therefore, if there exists a linear dependence between any of the solutions found, we cannot find among them even a single solution of the group $r = 1$, since without this condition there would not exist linear dependence of the particular solutions for the variable x_{i_1}. By similar considerations it is possible to show that no solutions of the group $r = 2$ may enter into linear dependence with the particular solutions, since x_{i_2}, entering into this group of solutions, does not enter by definition into any of the groups $r > 2$. Continuing these considerations further and eliminating step by step from the assumed linear dependence of the group of solutions $r = 1, 2, \ldots, k$, we prove by this the linear independence of all systems of the particular solutions found for the root $\lambda = \lambda_0$.

Considering now all distinct and non-equal roots of the characteristic equation, we obtain by this procedure the system of n particular solutions, the linear independence of which is obvious, if, following what has been proved, we direct our attention to the distinct factors of $e^{\lambda t}$ in the solutions, corresponding to distinct roots λ.

Note. It is possible to present many other methods of determining the particular solutions.

Without altering the above considerations, we may determine the polynomials A_{1j}, by the following formulae:

$$A_{1j_s} = \frac{\Delta_{i_1 \dots i_r\, j_1 \dots j_{r-1} j_s}}{D_{n-r}},$$

instead of (10),
where D_{n-r} is the highest common divisor of the rth-order minors.

Elementary Divisors

27[1]. To determine the highest common divisors of the minors of the characteristic determinant $\Delta(\lambda)$ elementary procedures have been developed, one of which we now consider.

By elementary transformations of the λ-matrix* are meant the following operations:

1) Interchange of two rows or two columns.

2) Multiplication of all elements of any row (column) by a single non-zero constant multiplier.

3) Addition of the elements of some row (column), multiplied by a single polynomial in λ, to the corresponding elements of another row (column).

These elementary transformations have the following properties which are important for our problem: if all i-th-order determinants of the λ-matrix, in particular the minors of the determinant $\Delta(\lambda)$, have as a factor the polynomial $\varphi(\lambda)$, all i-th-order determinants of each of the λ-matrices obtained from the initial one by elementary

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* That is a matrix, the elements of which are polynomials in λ.

transformations will have this factor. Now this is obvious for transformations 1 and 2. To prove that it is valid for transformation 3, let us add to the elements of the p-th column of the λ-matrix the corresponding elements of the q-th column, multiplied by the polynomial $\psi(\lambda)$. Each i-th-order determinant of the λ-matrix from which the p-th column is absent remains unchanged, as does each determinant of i-th order which includes both the p-th column and the q-th column. The i-th-order determinant including the p-th column but from which the q-th column is absent, takes the form

$$A \pm \psi B,$$

after the transformation,
where A and B are i-th-order determinants of the initial λ-matrix. From this the validity of the above assumption follows directly.

Elementary transformations cannot change the highest common divisor D_i of the i-th-order determinant of the initial λ-matrix. In accordance with the above, they may only increase the highest common divisor of the i-th-order determinant by some factor in the transformed λ-matrix, but they cannot increase this common divisor since the elementary operations are reversible, while the inverse transformation of the transformed λ-matrix (to the initial form) would then give a different expression for D_i.

The elementary transformations are of value since, as the highest common divisor of the determinant of the i-th-order λ-matrix is preserved, they may bring the latter to the so-called canonical form, for which it is not difficult to determine the highest common divisor of the determinant of arbitrary order.

If the first element $f(\lambda)$ of the λ-matrix does not vanish identically, and is not a factor of all the remaining elements, it is possible to obtain by elementary transformations a matrix, the first element of which does not vanish identically and will be of lower degree than $f(\lambda)$. In fact, we assume that in the first row there is located in the j-th column the element $f_1(\lambda)$, not divisible by $f(\lambda)$. If $f_1(\lambda)$ is of lower degree than $f(\lambda)$, this may be obtained by simple interchange of the first and j-th columns. In all other cases

$$f_1(\lambda) = q(\lambda) f(\lambda) + r(\lambda),$$

where $r(\lambda)$ is a polynomial of lower degree than $f(\lambda)$. Then, adding to the elements of the j-th column the elements of the first column, multiplied by $-q(\lambda)$, we obtain in the transformed λ-matrix the element $r(\lambda)$ in the first row and j-th column; if this element is put in place of the first, we obtain the required result.

These considerations extend even to the case where the element in the first column is not divisible by $f(\lambda)$.

Let now the element not divisible by $f(\lambda)$ be in the i-th row and the j-th column. Let the element of the first row and j-th column be $\psi(\lambda)f(\lambda)$. Adding now the elements in the first column, multiplied by $-\psi(\lambda)$, to the elements of the j-th column, we obtain zero in the first line and j-th column. Adding the j-th column to the first, we obtain $f(\lambda)$ as the first element, while in the second column and the i-th line will be an element not divisible by $f(\lambda)$, by the use of which we now know how to decrease the degree of the first element.

In this way we may always make the first element of the λ-matrix an element of lowest degree in λ not vanishing identically, if the initial λ-matrix does not consist entirely of zeros. If this element is not a common factor of the remaining elements, we repeat this process, each time reducing the degree of the first element so that after a finite number of operations the possibility of applying this procedure ceases (namely when the first element becomes the common factor of all the remaining ones).

By transformations 3 all elements of the first row and first column may be then brought to zero, except the first element E_1, and the quadratic matrix $\Delta(\lambda)$ of rank n under consideration takes the form

$$\begin{Vmatrix} E_1 & 0 & \dots & 0 \\ 0 & b_{22} & \dots & b_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & b_{n2} & \dots & b_{nn} \end{Vmatrix},$$

where $E_1(\lambda) \not\equiv 0$ is a divisor of all b_{rs}.

Continuing this process further on the matrix $|b_{ij}|$ and further matrices, we reduce the initial $\Delta(\lambda)$-matric to the form

$$\begin{Vmatrix} E_1 & 0 & \dots & 0 \\ 0 & E_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & E_n \end{Vmatrix},$$

since the rank of the initial matrix in the case of the characteristic determinant $\Delta(\lambda)$ is equal to n. In each polynomial $E_i(\lambda)$ the coefficient in the highest power of λ is made equal to unity. The polynomial $E_i(\lambda)$ by the process of transformation is a divisor of $E_j(\lambda)$ if $j > i$.

28. The highest common divisor of the i-th-order determinant of the last matrix is equal to

$$D_i(\lambda) = E_1(\lambda) \dots E_i(\lambda).$$

If the final matrix is obtained by elementary transformations of the $\Delta(\lambda)$-matrix, the roots of the polynomials $E_i(\lambda)$ are roots of the characteristic equation $\Delta(\lambda) = 0$, since the value of the determinant $\Delta(\lambda)$ is obviously proportional to the product

$$E_1(\lambda) \dots E_n(\lambda).$$

Denoting by

$$\lambda_1, \dots, \lambda_s$$

the distinct roots of the characteristic determinant $\Delta(\lambda)$, we can write the polynomial $E_i(\lambda)$ in elementary factors by the formula

$$E_i(\lambda) = (\lambda - \lambda_1)^{e_i^1} \dots (\lambda - \lambda_s)^{e_i^s} \qquad (i = 1, \dots, n).$$

The factors entering into the functions $E_i(\lambda)$

$$(\lambda - \lambda_j)^{e_i^j}$$

with the non-zero exponents e_i^j are termed elementary divisors. For each root λ_j the corresponding exponents e_i^j satisfy the inequality

$$e_i^j \geqslant e_{i-1}^j \qquad (i = 2, \dots, n),$$

since with reduction of $\Delta(\lambda)$ to the canonical form of the polynomial E_i each E_j is obtained as a divisor if $j > i$.

The elementary divisors of $\Delta(\lambda)$ define the structure of

the particular solutions for linear differential equations with constant coefficients. Now let us consider a solution corresponding to the root $\lambda = \lambda_0$, for which the elementary divisors have the form

$$(\lambda - \lambda_0)^{e_i}.$$

The r-th-order minor is a determinant of order $l = n - r$. We see from this that the multiplicity μ_r of the root $\lambda = \lambda_0$ in the highest common divisor D_{n-r} of the minors of order will be:

$$\mu_r = e_1 + \ldots + e_{n-r},$$

and, consequently, the difference $\mu_{r-1} - \mu_r$, defined in section 26 as the number of particular solutions in the group corresponding to the root $\lambda = \lambda_0$, is

$$\mu_{r-1} - \mu_r = e_{n-r+1}.$$

By this we establish that the exponents e_i^j of the elementary divisors are equal to the numbers of solutions in the group corresponding to the root λ_j.

29. Example. Consider a system of differential equations

$$\frac{dx}{dt} = x', \quad \frac{dx'}{dt} = -x - 2x',$$

$$\frac{dy}{dt} = y', \quad \frac{dy'}{dt} = -y - 2y' + \mu(x + x').$$

If desired this example may be modelled by two mechanical systems with the indicated forces, in which the first system applies the forces $\mu(x + x')$ to the second system through some servo-system.

We shall follow the methods of sections 25, 26.

Rule 1. It is first necessary to determine the elementary divisors of the characteristic matrix $\Delta(\lambda)$.

Rule 2. For some root $\lambda = \lambda_0$, beginning with the minor of the highest order k, which does not vanish for $\lambda = \lambda_0$, determine the chain of minors (inclusive of sign)

$$\Delta_{i_1 \ldots i_r, j_1 \ldots j_r}$$

with some common indices for $r = 1, \ldots, k$ such that in

passing from r to $r-1$ the multiplicity of the root $\lambda=\lambda_0$ is increased for these minors precisely by the magnitude

$$e_{n-r+1},$$

and further for each of the values $r=1, \ldots, k$ determine the ne_{n-r+1} polynomials

$$A_{1j_s}=\frac{\Delta_{i_1 \ldots i_r,\, j_1 \ldots j_{r-1} j_s}}{(\lambda-\lambda_0)^{\mu r}}, \quad A_{p+1,\,j}=\frac{A_{1j}^{(p)}}{p!}$$

$$(p=1, \ldots, e_{n-r+1}-1;\ s=r, \ldots, n);$$

equate to zero all A_{1j_s} for $s<r$. From these polynomials; (putting in them $\lambda=\lambda_0$), determine the numbers A and determine the solution having secular terms of the highest degree $m=e_{n-r+1}-1$ of the group (r)

$$x_i=\left(A_{1i}\frac{t^m}{m!}+\ldots+A_{m+1,\,i}\right)e^{\lambda t} \qquad (\lambda=\lambda_0).$$

To determine the elementary divisors of the matrix

$$\Delta(\lambda)=\begin{Vmatrix} -\lambda & 1 & 0 & 0 \\ -1 & -2-\lambda & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ \mu & \mu & -1 & -2-\lambda \end{Vmatrix}$$

we add the second column, multiplied by λ, to the first and transpose the first and second columns

$$\begin{Vmatrix} 1 & 0 & 0 & 0 \\ -2-\lambda & -(\lambda+1)^2 & 0 & 0 \\ 0 & 0 & -\lambda & 1 \\ \mu & \mu(\lambda+1) & -1 & -2-\lambda \end{Vmatrix}.$$

All elements of the first column except the first may be brought to zero by elementary operations on the rows. Assuming this to have been carried out, we multiply the third row by $\lambda+2$ and add it to the fourth row, after which we transpose the second and third rows while the fourth column is transposed with the second, placing the third column in place of the fourth and the second in place of the third:

$$\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\lambda \\ 0 & 0 & -(\lambda+1)^2 & 0 \\ 0 & 0 & \mu(\lambda+1) & -(\lambda+1)^2 \end{Vmatrix}.$$

All elements of the second row except the second are made to vanish. In the non-canonical part of the matrix, $\lambda+1$ is the common factor of all its elements. Assuming μ to be different from zero, the fourth row, multiplied by $\frac{\lambda+1}{\mu}$, is added to the third and their positions interchanged:

$$\begin{Vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \lambda+1 & 0 \\ 0 & 0 & 0 & (\lambda+1)^3 \end{Vmatrix}.$$

Consequently, the elementary divisors of $\Delta(\lambda)$ will be:

$$(\lambda+1),\ (\lambda+1)^3.$$

The characteristic equation $\Delta(\lambda)=0$ has a single root $\lambda=-1$ of multiplicity four, to which, in accordance with the structure of the elementary divisors, as discussed above, there will correspond two groups of particular solutions: one with a single and the other with three particular solutions.

It is to be understood that, in practical calculations, as soon as the factor E_α is determined, it is necessary to return the matrix immediately to its non-canonical form.

We shall now determine the particular solutions. In accordance with the canonical form found for the matrix $\Delta(\lambda)$, we conclude that all its minors of first order will vanish for $\lambda=-1$ while some of the second-order minors will be non-vanishing for $\lambda=-1$. Among the minors of $\Delta(\lambda)$ we determine the chain of the highest common factors (rule 2), for example

$$\Delta_{14,\,23}=1,\quad \Delta_{1,\,3}=-\mu(\lambda+1).$$

The group of particular solutions $r=2$ corresponding to the elementary divisor $\lambda+1$ is calculated from rule 2. The numbers A_{1j} are

$$A_{11}=-1,\ A_{12}=1,\ A_{13}=0,\ A_{14}=0;$$

from which we see that the group $r=2$ consists of the particular solution

$$x=-e^{-t},\quad x'=e^{-t},\quad y=0,\quad y'=0.$$

To determine the group of particular solutions $r=1$,

corresponding to the elementary divisor $(\lambda+1)^3$, we follow rule 2; the polynomials $A_{i,j}$ for this group have the form

$$A_{11} = -(\lambda+2)(\lambda+1),\ A_{12} = \lambda+1,$$
$$A_{13} = -\mu,\quad A_{14} = -\mu\lambda.$$

From this we obtain by differentiation:

$$A_{21} = -2\lambda-3,\quad A_{22} = 1,\quad A_{23} = 0,\quad A_{24} = -\mu,$$
$$A_{31} = -1,\quad A_{32} = 0,\quad A_{33} = 0,\quad A_{34} = 0.$$

Substituting in these polynomials $\lambda = -1$, we find the constants A_{kj} and form the solution of the group

$$x = (-t-1)e^{-t},\ x' = te^{-t},$$
$$y = -\mu\frac{t^2}{2}e^{-t},\quad y' = \left(\mu\frac{t^2}{2} - \mu t\right)e^{-t}.$$

The other two derivative particular solutions for the group $r=1$ are obtained by differentiating the polynomial factors in the solution already found:

$$x = -e^{-t},\quad x' = e^{-t},\quad y = -\mu te^{-t},\quad y' = (\mu t - \mu)e^{-t},$$
$$x = 0,\quad x' = 0,\quad y = -\mu e^{-t},\quad y' = \mu e^{-t}\ ^{1}).$$

*

Canonical Form of the First Approximation

30[18L]. For the systems of differential equations with constant coefficients under study

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n$$

we shall seek n independent integrals of the form

$$U = y_1x_1 + \ldots + y_nx_n,$$

---

* It is necessary to remark that if the proposed example is solved by the frequently employed method of arbitrary elimination of variables, bringing the system to the equation $\Delta\left(\frac{d}{dt}\right)y = 0$, in which the degree of the operator $\frac{d}{dt}$ denotes the order of the derivatives, a necessarily erroneous result is obtained, giving the existence of a particular solution with secular term t^3, which is, in fact, absent from the problem.

where y_i are certain functions of t. The equation to be satisfied by the functions y_i are obtained by differentiation

$$\frac{dU}{dt} = \sum_i x_i \frac{dy_i}{dt} + \sum_i y_i (p_{i1}x_1 + \ldots + p_{in}x_n) =$$
$$= \sum_i x_i \left(\frac{dy_i}{dt} + p_{1i}y_1 + \ldots + p_{ni}y_n \right) \equiv 0.$$

Since this identity should be valid for arbitrary initial values of the variables x_i, while for the initial instant we are free to take any time, including that under consideration, then

$$\frac{dy_i}{dt} + p_{1i}y_1 + \ldots + p_{ni}y_n = 0.$$

This system of differential equations is termed the adjoint to the initial system. Its characteristic equation has the form

$$D(\mu) = \| p_{ji} + \delta_{ji}\mu \| = 0.$$

In other words, the characteristic determinant of the adjoint system is obtained from the initial $\Delta(\lambda)$ by putting $-\mu$ in place of λ and by interchange of rows and columns and vice versa. Therefore the characteristic roots of the adjoint system μ_s differ only in sign from the characteristic roots λ_s of the initial system.

Let

$$(\mu + \lambda_s)^{n_s} \quad (s = 1, \ldots, k)$$

represent all the elementary divisors of the characteristic matrix $D(\mu)$ for the adjoint system, under the assumption that each root is repeated as many times as the number of groups of solutions corresponding to it. The degree n_s of the corresponding elementary divisor denotes, as we know, the number of solutions in the group corresponding to the root $\mu = -\lambda_s$; the numbers n_s include all the exponents of the elementary divisors of the matrix $\Delta(\lambda)$. Let the group of particular solutions of the adjoint system for the divisor (s) be

$$y_i^{(s)} = \left(A_{1i}^{(s)} \frac{t^m}{m!} + \ldots + A_{mi}^{(s)} \right) e^{-\lambda_s t} \quad (m = 0, 1, \ldots, n_s - 1),$$

where $A_{ki}^{(s)}$ are constants, the determination of which has been defined. Substituting these particular solutions y_i in

the expression for the required integral U, we obtain n_s successive integrals of the system of variational equations:

$$U^{(s)} = \left(z_1^{(s)} \frac{t^m}{m!} + \ldots + z_{m+1}^{(s)}\right) e^{-\lambda_s t} \quad (m = 0, 1, \ldots, n_s - 1),$$

where $z_r^{(s)}$ are linear forms with regard to the variables $x_1, \ldots, x_n$ with constant coefficients:

$$z_r^{(s)} = \sum_i A_{ri}^{(s)} x_i \qquad (r = 1, \ldots, n_s). \tag{11}$$

All $n = n_1 + \ldots + n_k$ integrals which we obtain, by giving s all values from 1 to k inclusive, will be linearly independent. Thus the n linear forms

$$z_r^{(s)} \ (r = 1, \ldots, n_s; \quad s = 1, \ldots, k),$$

will be independent and consequently may be taken as new variables in place of $x_1, \ldots, x_n$. Differentiating the integral $U^{(s)}$ of highest degree $m = n_s - 1$, we have:

$$e^{-\lambda_s t}\left[\left(\frac{dz_1^{(s)}}{dt} - \lambda_s z_1^{(s)}\right)\frac{t^{n_s-1}}{n_s-1!} + \left(\frac{dz_2^{(s)}}{dt} - \lambda_s z_2^{(s)} + z_1^{(s)}\right)\frac{t^{n_s-2}}{n_s\ 2!} + \right.$$
$$\left. + \ldots + \left(\frac{dz_{n_s}^{(s)}}{dt} - \lambda_s z_{n_s}^{(s)} + z_{n_s-1}^{(s)}\right)\right] = 0,$$

from which

$$\begin{cases} \dfrac{dz_1^{(s)}}{dt} = \lambda_s z_1^{(s)}, \\ \dfrac{dz_j^{(s)}}{dt} = \lambda_s z_j^{(s)} - z_{j-1}^{(s)} \qquad (s = 1, \ldots, k; \ j = 2, \ldots, n_s). \end{cases} \tag{12}$$

Thus, using the linear non-singular* transformation with constant coefficients (11) the system of variational differential equations is transformed to the canonical form (12).

The canonical variables $z_j^{(s)}$ divide the space $(x_1, \ldots, x_n)$ into k sub-spaces $(z_1^{(s)}, \ldots, z_{n_s}^{(s)})$, where $s = 1, \ldots, k$ in the sense that the initial perturbation of any single variables $z_j^{(s)}$ does not affect the variables $z_\alpha^{(\beta)}$ for β different from s, and for $\beta = s$, $\alpha < j$.

---

* By linear non-singular transformation is understood that for which the determinant of the coefficients is non-vanishing.

When complex characteristic roots λ_s exist the canonical variables $z_j^{(s)}$ corresponding to them will also be complex. If we want only real variables, a supplementary substitution of variables should be carried out.

Let us assume that the conjugate roots

$$\lambda_1 = \lambda + \mu\sqrt{-1}, \quad \lambda_2 = \lambda - \mu\sqrt{-1}$$

correspond to the following values of z:

$$z_j^{(1)} = u_j + v_j\sqrt{-1},$$
$$z_j^{(2)} = u_j - v_j\sqrt{-1} \qquad (j = 1, \ldots, n_1 = n_2).$$

It is possible in place of $z_j^{(1)}$, $z_j^{(2)}$ to adopt as the new variables the quantities u_j, v_j, which will be linear forms of the quantities x_s with constant real coefficients. Differential equations satisfying u_j, v_j are obtained from (12) by separation of the real and the imaginary parts:

$$\frac{du_1}{dt} = \lambda u_1 - \mu v_1,$$
$$\frac{dv_1}{dt} = \lambda v_1 + \mu u_1,$$
$$\frac{du_j}{dt} = \lambda u_j - \mu v_j - u_{j-1},$$
$$\frac{dv_j}{dt} = \lambda v_j + \mu u_j - v_{j-1} \qquad (j = 2, \ldots, n_1).$$

We obtain such groups of equations for each pair of complex-conjugate roots. For real roots we have a group of the form (12).

31. Example. Consider the equations

$$\frac{dx_1}{dt} = -x_4 - x_6, \quad \frac{dx_4}{dt} = x_1 + x_4 - x_5,$$
$$\frac{dx_2}{dt} = x_2 + x_3, \quad \frac{dx_5}{dt} = x_1 + x_5 + x_6,$$
$$\frac{dx_3}{dt} = -x_2 + x_3, \quad \frac{dx_6}{dt} = x_1 + x_4 - x_5 + 2x_6.$$

To determine the canonical variables $z_j^{(s)}$, let us write the matrix of the adjoint system

$$D(\mu)=\begin{Vmatrix} \mu & 0 & 0 & 1 & 1 & 1 \\ 0 & 1+\mu & -1 & 0 & 0 & 0 \\ 0 & 1 & 1+\mu & 0 & 0 & 0 \\ -1 & 0 & 0 & 1+\mu & 0 & 1 \\ 0 & 0 & 0 & -1 & 1+\mu & -1 \\ -1 & 0 & 0 & 0 & 1 & 2+\mu \end{Vmatrix}.$$

The elementary factors of this matrix are

$$\mu^2+2\mu+2, \quad (\mu^2+2\mu+2)^2.$$

Consequently the characteristic equation of the adjoint system $D(\mu)=0$ has two complex-conjugate roots $\mu=-1+i$ and $\mu=-1-i$, each of multiplicity three; the characteristic equation of the given system $\Delta(\lambda)=0$ will therefore have roots $\lambda=1-i$ and $\lambda=1+i$ of the same multiplicity.

To determine the constants $A_{kj}^{(s)}$, entering into the expressions of the canonical variables $z_k^{(s)}$ and the solutions of the adjoint system, we consider the chain of minors:

$$D_{42,58}=-\mu(2+\mu), \quad D_{4,5}=\mu(2+\mu)(\mu^2+2\mu+2),$$

satisfying the rule of section 29 for an arbitrary root of the characteristic equation of the adjoint system $D(\mu)=0$.

Initially we determine the constants A_{kj}, connected in the well-known sense with the simple elementary divisor $(\mu+1+i)$. From the formula

$$A_{1j}=D_{42,5j}$$

we determine the polynomials

$$A_{11}=0, \quad A_{12}=\mu(1+\mu)(2+\mu), \quad A_{13}=-\mu(2+\mu),$$
$$A_{14}=0, \quad A_{15}=0, \quad A_{16}=0,$$

from which for the given root $\mu=-1-i$ we have:

$$A_{12}=2i, \quad A_{13}=2,$$

and consequently, from formula (11)

$$z_1^{(1)}=2(x_3+ix_2).$$

We now determine the canonical variables of the second

group of the same root. Since all first-order minors of the determinant $D(\mu)$ have the common factor $\mu^2+2\mu+2$ we may determine the polynomial A_{1j}, following the note in section 26, from the formulae

$$A_{1j}=\frac{D_{4j}}{\mu^2+2\mu+2}.$$

We have

$$A_{11}=-(2+\mu)^2, \quad A_{12}=0, \quad A_{13}=0,$$
$$A_{14}=(1+\mu)\cdot(\mu^2+2\mu+2), \quad A_{15}=\mu(2+\mu), \quad A_{16}=-2(1+\mu).$$

From formula (9)

$$A_{21}=-2(2+\mu), \quad A_{22}=0, \quad A_{23}=0,$$
$$A_{24}=(\mu^2+2\mu+2)+2(1+\mu)^2, \quad A_{25}=2(1+\mu), \quad A_{26}=-2.$$

Substituting in these polynomials the value of the root $\mu=-1-l$, we have

$$A_{11}=2l, \quad A_{12}=0, \quad A_{13}=0,$$
$$A_{14}=0, \quad A_{15}=-2, \quad A_{16}=2l,$$
$$A_{21}=-2+2l, \quad A_{22}=0, \quad A_{23}=0,$$
$$A_{24}=-2, \quad A_{25}=-2l, \quad A_{26}=-2,$$

from which the canonical variables of this group are determined from formulae (11)

$$z_1^{(2)}=2l(x_1+x_6+lx_5), \quad z_2^{(2)}=-2(x_1+x_6+lx_5)+2l(x_1+ix_4).$$

The canonical variables for the conjugate root are obtained by passing to the conjugate forms $\bar{z}_1^{(1)}$, $\bar{z}_1^{(2)}$, $\bar{z}_2^{(2)}$.

32[17L]. Particular solutions of the equations in section 24 or the solutions easily obtained from their substitution of the canonical equations (12) in equations (11),

$$z_j^{(s)}=\varphi_j^{(s)}e^{\lambda_s t} \qquad (\varphi_j^{(s)}(t)\text{ — polynomials}$$

lead to immediate conclusions as to the stability in the first approximation.

If among the roots $\lambda_s=\alpha_s+i\beta_s$ of the characteristic equation $\Delta(\lambda)=0$ there is at least one root with positive real part, the unperturbed motion $(x_1=0, \ldots, x_n=0)$ is unstable in the first approximation. If all these roots have

negative real parts, the unperturbed motion is stable in the first approximation. If a part of the roots have real parts equal to zero and the other roots negative real parts, the unperturbed motion will be stable or unstable in the first approximation, according to whether all elementary divisors corresponding to the zero and purely imaginary roots of the determinant $\Delta(\lambda)$ are simple (with exponents e_i^s, not exceeding 1) or among them is found at least one not simple (with exponent e_i^s greater than one).

As an exercise it is possible to prove all this also by a direct method, by considering the function

$$V = \sum_s (\alpha_s - \varepsilon)\,[z_1^{(s)}\bar{z}_1^{(s)} + \\ + (\alpha_s - \varepsilon)^2 z_2^{(s)}\bar{z}_2^{(s)} + \ldots + (\alpha_s - \varepsilon)^{2(n_s-1)} z_{n_s}^{(s)}\bar{z}_{n_s}^{(s)}],$$

where ε is a certain real constant different from α_s. The total derivative with regard to t of this function V, by reason of the canonical equations (12) is

$$V' = 2\varepsilon V + \sum_s (\alpha_s - \varepsilon)^2 [2z_1^{(s)}\bar{z}_1^{(s)} + 2(\alpha_s - \varepsilon)^2 z_2^{(s)}\bar{z}_2^{(s)} + \ldots + \\ + 2(\alpha_s - \varepsilon)^{2(n_s-1)} z_{n_s}^{(s)}\bar{z}_{n_s}^{(s)} - \\ - (\alpha_s - \varepsilon)(z_1^{(s)}\bar{z}_2^{(s)} + \bar{z}_1^{(s)}z_2^{(s)}) - \ldots - \\ - (\alpha_s - \varepsilon)^{2n_s-3}(z_{n_s-1}^{(s)}\bar{z}_{n_s}^{(s)} + \bar{z}_{n_s-1}^{(s)}z_{n_s}^{(s)})].$$

The discriminant of the Hermitian quadratic form, in square brackets under the summation sign, is

$$\begin{vmatrix} 2 & -1 & \ldots & 0 \\ -1 & 2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 2 \end{vmatrix},$$

if, for simplicity, we take as the variables $(\alpha_s - \varepsilon)^k z_{k+1}^{(s)}$.

The principal diagonal minors Δ_r of this discriminant are all positive*, therefore the forms in square brackets

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* The expansion of Δ_r, if $r \geqslant 3$, in elements of the last column leads to the relationship $\Delta_r - \Delta_{r-1} = \Delta_{r-1} - \Delta_{r-2}$, from which, after calculation of $\Delta_1 = 2$, $\Delta_2 = 3$, we obtain $\Delta_r = r + 1$.

will each be positive-definite with regard to its variables $z_1^{(s)}, \ldots, z_{n_s}^{(s)}$, and the entire sum will thus be positive-definite with regard to the variables $x_1, \ldots, x_n$.

If among the roots $\lambda_s = \alpha_s + i\beta_s$ of the characteristic equation $\Delta(\lambda) = 0$ there is at least one, taken as the root λ_1, with positive real part ($\alpha_1 > 0$), then, taking ε equal to some positive number not exceeding the smallest positive real part of the root, we note that the function V for such choice of ε satisfies all conditions of Lyapunov's second theorem on instability since with choice of variables $x_1, \ldots, x_n$ in accordance with the conditions $z_j^{(s)} = 0 (s > 1)$ the values of the function V will be positive. The unperturbed motions will therefore be unstable.

If all roots $\lambda_s = \alpha_s + i\beta_s$ have negative real parts $\alpha_s < 0$, then, taking ε in accordance with the inequality $\alpha_s \leqslant \varepsilon \leqslant 0$, we note that the function V will be negative-definite and its derivative positive-definite. From Lyapunov's theorem the unperturbed motion will be stable and slightly perturbed motions will tend to it asymptotically.

The freedom in choosing ε shows that in constructing the function V, an exact knowledge of the roots λ_s and the canonical variables $z_j^{(s)}$ is not necessary to settle the question of stability or instability in the first approximation. The practical value of this will be shown below.

Hurwitz's Theorem

33. In many problems, where it is necessary to find only the conditions which must be satisfied by the coefficients p_{rs} to ensure stability or unstability of unperturbed motion in the first approximation, there is no sense in calculating the roots of the characteristic equation $\Delta(\lambda) = 0$; it is sufficient to know the signs of their real parts.

The necessary and sufficient conditions for negative real parts of all roots of a polynomial with real coefficients were first found by Routh. A better-known condition was proposed by Hurwitz.

The polynomial

$$f(z) = z^n + a_1 z^{n-1} + \ldots + a_n$$

with real coefficients a is termed a Hurwitz polynomial if

all its roots have negative real parts. In the plane of the complex variable $z = x + iy$, all roots of the Hurtwitz polynomial lie within the region Φ, bounded by the imaginary y-axis and a left semi-circle of sufficiently large radius R with centre at the origin

Let

$$f^*(z) = z^n - a_1 z^{n-1} + \ldots + (-1)^n a_n = (-1)^n f(-z).$$

Let us consider the polynomial of degree $n+1$

$$F_0 = (z+c) f(z),$$

where c is a certain positive constant; if $f(z)$ is a Hurwitz polynomial, obviously the polynomial F_0 will be also.

The polynomial of degree $(n+1)$

$$F_\mu = (z+c) f(z) + \mu z f^*(z)$$

varies from F_0 to the polynomial

$$F_1 = (z+c) f(z) + z f^*(z),$$

as μ varies continuously from 0 to 1.

If $f(z)$ is a Hurwitz polynomial F_μ, with continuous variation of the parameter μ from 0 to 1, may cease to be a Hurwitz polynomial if for any value of the parameter μ in the interval (0, 1) at least one of the roots of the polynomial F_μ passes outside the boundary of the region Φ. But for the given values of the parameter μ, the polynomial F_μ has no roots on the left semi-circle of sufficiently large radius R such that, substituting on it $z = Re^{i\vartheta}$, we have:

$$F_\mu = (1+\mu) R^{n+1} e^{i(n+1)\vartheta} + \ldots$$

Therefore, if some roots F_μ leave the region Φ, this may occur only on the passage of the root through the imaginary axis. But this also cannot occur for $0 \leqslant \mu \leqslant 1$. In fact, expanding the polynomial $f(z)$ in linear factors corresponding to its roots α_k, and the polynomial $f^*(z)$ in linear factors corresponding to the roots $\alpha_k^* = -\bar{\alpha}_k$, we verify directly that on the y-axis (if $f(z)$ is a Hurwitz polynomial)

$$\left| \frac{f(z)}{f^*(z)} \right| = \frac{\gamma_1 \gamma_2 \ldots \gamma_n}{\gamma_1^* \gamma_2^* \ldots \gamma_n^*} = 1,$$

where

$$\rho_k = |z - \alpha_k|, \quad \rho_k^* = |z - \alpha_k^*| = |z + \bar{\alpha}_k|,$$

since on the y-axis $(z = iy)$ we have $\rho_k = \rho_k^*$.

Under our conditions $c > 0$ and $0 \leqslant \mu \leqslant 1$, the polynomial F_μ cannot vanish on the y-axis $(z = iy)$ since in this case we would have

$$1 = \left| \frac{(z+c) f(z)}{\mu z f^*(z)} \right| = \left| \frac{z+c}{\mu z} \right|$$

or

$$c^2 + (1 - \mu^2) y^2 = 0,$$

which is impossible.

Thus, if $f(z)$ is a Hurwitz polynomial, the polynomial

$$F_1 = (z+c) f(z) + z f^*(z) \qquad (c > 0).$$

will also be a Hurwitz polynomial.

The reverse process shows that for the Hurwitz polynomial F_1 of degree $n+1$, the Hurwitz polynomial $f(z)$ of degree n can be formed. In fact, consider the polynomial

$$\Phi_\mu = (z - 2a_1) f(z) - \mu z f^*(z) \qquad (a_1 > 0),$$

where the parameter μ varies from 0 to 1. If $f(z)$ is a Hurwitz polynomial, the polynomial $\Phi_0 = (z - 2a_1) f(z)$ will have n roots in the left half-plane of the variable z and a single root $z = 2a_1 > 0$ in the right half-plane. In the interval of variation of μ none of the polynomials Φ_μ can have a purely imaginary root (this is proved exactly the same as for the polynomial F_μ). As $\mu \to 1$, the polynomial

$$\Phi_\mu = (1 - \mu) z^{n+1} - (1 - \mu) a_1 z^n + [a_2 (1 - \mu) - 2a_1^2] z^{n-1} + \ldots$$

will have two real roots at infinity of the form

$$\pm a_1 \sqrt{\frac{2}{1 - \mu}} (1 + \ldots),$$

where the omitted terms tend to zero for μ tending to unity. These roots are of opposite signs; therefore the polynomial of degree $(n-1)$

$$\Phi_1 = (z - 2a_1) f(z) - z f'(z)$$

has all its roots in the left half-plane; and thus if $f(z)$ is a Hurwitz polynomial, the polynomial Φ_1 will then also be a Hurwitz polynomial.

It is now possible to describe Hurwitz's theorem. From the coefficients of the polynomial $f(z)$ we form the table

$$\begin{Vmatrix} a_1 & 1 & 0 & 0 & 0 & 0 \dots \\ a_3 & a_2 & a_1 & 1 & 0 & 0 \dots \\ a_5 & a_4 & a_3 & a_2 & a_1 & 1 \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{Vmatrix},$$

in which all $a_m = 0$ if $m > n$. On its principal diagonal are successively $a_1, a_2, a_3, \dots$ The principal diagonal minors of this table will be denoted by

$$\Delta_1 = a_1, \quad \Delta_2 = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix}, \qquad \Delta_n = a_n \Delta_{n-1}.$$

Hurwitz's Theorem. The necessary and sufficient condition for the polynomial $f(z)$ to have all roots with negative real parts is that all the following inequalities are satisfied:

$$\Delta_1 > 0, \quad \Delta_2 > 0, \quad \dots, \quad \Delta_n > 0.$$

Proof. Let $c = 2a > 0$. Then

$$F_1 = 2\,[z^{n+1} + az^n + (aa_1 + a_2) z^{n-1} + aa_2 z^{n-2} + \dots].$$

Let us denote by Δ_n the Hurwitz determinant for the polynomial $f(z)$ and by D_n the Hurwitz determinant for the polynomial $\frac{1}{2} F_1$. We have:

$$D_n = \begin{vmatrix} a & 1 & 0 & 0 & \dots \\ aa_2 & aa_1 + a_2 & a & 1 & \dots \\ aa_4 & aa_3 + a_4 & aa_2 & aa_1 + a_2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = a^n \Delta_{n-1}.$$

In fact we remove the factor a from odd columns, after which each of these columns is subtracted from the even column with index higher by unity; removing the factor a from the even columns as well, we have:

$$D_\nu = a^\nu \begin{vmatrix} 1 & 0 & 0 & \dots \\ a_2 & a_1 & 1 & \dots \\ a_4 & a_3 & a_2 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = a^\nu \Delta_{\nu-1}.$$

The necessary and sufficient condition that F_1 be a Hurwitz polynomial is that $f(z)$ is a Hurwitz polynomial.

Let us now assume that the Hurwitz criterion is valid for the polynomials of degrees $\leqslant n$.

If $\Delta_\nu > 0$ $(\nu = 1, \dots, n)$, $f(z)$ is a Hurwitz polynomial and, consequently, F_1 will a Hurwitz polynomial, and for it there will necessarily exist the condition

$$D_r > 0 \qquad (r = 1, \dots, n+1).$$

If $D_r > 0$ $(r = 1, \dots, n+1)$, $\Delta_s > 0$ $(s = 1, \dots, n)$ and so $f(z)$ will be a Hurwitz polynomial in accordance with the hypothesis on the validity of the Hurwitz criterion for the polynomials of degree n. But the condition that $f(z)$ is a Hurwitz polynomial is sufficient for the polynomial $F_1(z)$ also to be a Hurwitz polynomial.

Thus, if the Hurwitz criterion is valid for polynomials of degree n, it will be valid $(D_1 = a > 0)$ also for the polynomials of degree $n+1$, as each Hurwitz polynomial of degree $n+1$ may be formed from some Hurwitz polynomial of degree n, in the way that the polynomial F_1 was formed from $f(z)$. For polynomials of the first and second degree the Hurwitz criterion is obviously valid; by the method of mathematical induction we find that it will be always valid.

Example[2]. The question of the stability of vertical rotation of a rigid body with a single fixed point (in Lagrange's case) with respect to the mutation angle ϑ leads to study of the differential equation

$$\left(\frac{du}{dt}\right)^2 = (\alpha - au)(1 - u^2) - (\beta - br_0 u)^2 = f(u),$$

in which $u = \cos\vartheta$ and α, β, a, b are certain constants, a and b positive; r_0 is the projection on the z-axis of the initial angular velocity of the rotating rigid body. For the mechanical problem all roots of the polynomial $f(u)$ are real while the values of the variable u lie in a closed interval between the smallest roots of the polynomial $f(u)$.

Let us find the condition for which the roots of the polynomial $f(u)$ will be greater than $1-\delta$, where δ is an arbitrarily assigned small positive magnitude.

Consider the polynomial

$$F(z) = -f(1-\delta-z) = az^3 + a_1z^2 + a_2z + a_3.$$

Its roots will all be negative if the roots of the polynomial $f(u)$ are greater than $1-\delta$; and conversely, if all roots of the polynomial $F(z)$ are negative, all roots of the polynomial $f(u)$ will be greater than $1-\delta$.

The conditions for all roots of the polynomial $F(z)$ to be negative are obtained from Hurwitz's theorem in the form of inequalities

$$a_1 > 0, \quad a_1a_2 - aa_3 > 0, \quad a_3 > 0.$$

For the case of greatest practical interest $\theta_0 = 0$, $\theta_0' > 0$ or $\beta - br_0 = 0$, $\theta_0'^2 = \alpha - a$, these inequalities will have the following explicit form if the implicit expressions for the coefficients a_1, a_2, a_3 are used:

$$b^2r_0^2 - 2a + \theta_0'^2 + 3a\delta > 0,$$

$$-2(b^2r_0^2 - 2a + \theta_0'^2)\theta_0'^2 + \delta[2(b^2r^2 - 2a + \theta_0'^2)^2 - 4a\theta_0'^2] + \\ + \delta^2 8a(b^2r_0^2 - 2a + \theta_0'^2) + 8a^2\delta^3 > 0,$$

$$-2\theta_0'^2 + \delta(b^2r_0^2 - 2a + \theta_0'^2) + a\delta^2 > 0.$$

We shall be interested in the smallest value of δ for which these three inequalities are valid; this value will be the lowest upper bound for the function $1-u$. If the stability reserve $s = b^2r_0^2 - 2a > 0$ is taken satisfying the inequality

$$(s + \theta_0'^2)^2 - 4a\theta_0'^2 \geqslant 0,$$

then for

$$\delta = \frac{2\theta_0'^2}{s + \theta_0'^2}$$

all three inequalities will be satisfied. And consequently, for such a stability reserve they will remain satisfied by continuity also for some smaller value of δ. Therefore for the given value of the stability reserve s

$$1 - u < \frac{2\theta_0'^2}{s + \theta_0'^2}.$$

Example. Let us consider the case where in the equations of perturbed motion

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n \qquad (s = 1, \ldots, n)$$

the coefficients p_{sr} are continuous bounded functions of the real parameter α. Differing unperturbed motions may correspond to differing values of α. The α-axis corresponds to the sequence of unperturbed motions considered ($x_1 = 0, \ldots, x_n = 0$) in the space of variables ($\alpha, x_1, \ldots, x_n$). Those points of the α-axis where the characteristic equation

$$\Delta(\lambda) = \|p_{sr} - \delta_{sr}\lambda\| = (-1)^n(\lambda^n + a_1\lambda^{n-1} + \ldots + a_n) = 0$$

has all roots with negative real parts correspond to asymptotically stable unperturbed motions; for this the Hurwitz determinants must be positive:

$$\Delta_1 = a_1 > 0, \quad \Delta_2 = \begin{vmatrix} a_1 & 1 \\ a_3 & a_2 \end{vmatrix} > 0, \ \ldots, \ \Delta_n = a_n\Delta_{n-1} > 0.$$

The determinants Δ_s will be certain functions of α. If the unperturbed motion corresponding to $\alpha = \alpha_0$ is asymptotically stable, with further change in the parameter α in any direction beginning from the value of α_0, stability may be lost either when at least one of the roots of the characteristic equation vanishes, or when two roots become purely imaginary; in the former case the coefficient a_n vanishes, in the second Δ_{n-1} vanishes. The first statement does not require proof. To prove the second let us divide the characteristic polynomial into sums of terms with even and odd powers

$$(-1)^n\Delta(\lambda) = (\lambda^n + a_2\lambda^{n-2} + \ldots) + (a_1\lambda^{n-1} + a_3\lambda^{n-3} + \ldots).$$

The total number of terms in the first of the expressions in parentheses (say, $\mu + 1$) and the number of terms in the second parentheses (say, $\nu + 1$) is equal to the number $n + 1$ of all terms of the polynomial $(-1)^n\Delta(\lambda)$. From this

$$\mu + \nu = n - 1.$$

Let us consider the polynomials

$$\varphi(z) = z^\mu + a_2z^{\mu-1} + \ldots, \qquad \psi(z) = a_1z^\nu + a_3z^{\nu-1} + \ldots$$

If the polynomial $\Delta(\lambda)$ has a purely imaginary root $\lambda = i\beta$, then the polynomials $\varphi(z)$ and $\psi(z)$ will have, from their derivation, at least one common root, namely $z = -\beta$. But the polynomials φ and ψ have a common root when and only when their resultant

$$R = \pm\Delta_{n-1}$$

vanishes.

Let the point P on the α-axis, where $\alpha = \alpha_0$, correspond to asymptotically stable unperturbed motion and, consequently, at that point all Δ_s are positive. With further increase in α the determinants Δ_s will change in some manner; a_n or Δ_{n-1} will necessarily vanish first. If, for the first violation of Hurwitz's inequalities, a_n vanishes, then for the corresponding point M the equation of perturbed motion will have by reason of the canonical form of equations (12) at least one linear integral or new sequence of "equilibria", passing through M, since $\Delta(\lambda)$ has then at least one zero root. If, for the first violation of Hurwitz's conditions, Δ_{n-1} vanishes, then in the region corresponding to point C we will have a single-parameter (at least) sequence of periodic motions, corresponding to the quadratic integral of the equations of perturbed motion, necessary by reason on the canonical form of equations (12), for a pair of imaginary roots of $\Delta(\lambda)$*.

From the above we should also conclude that if with change of α from α_0 to some value α_1 neither a_n nor Δ_{n-1} vanish, the unperturbed motion corresponding to $\alpha = \alpha_1$ will be asymptotically stable.

34[19, 20L]. From section 32, Hurwitz's inequalities ($\Delta_s > 0$, $s = 1, \ldots, n$), presented for the characteristic polynomial $f(z) \equiv \Delta(z)$, give the necessary and sufficient conditions for asymptotic stability.

Many algebraic forms of Hurwitz's theorem exist. We shall not consider them here; we note that similar

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* The question of the existence of periodic motion when the equations of perturbed motion are not confined to linear terms was studied by Kuz'min[3], while the question of the behaviour of a trajectory in the region of points of bifurcation M, and at branch points of periodic orbits C, was considered by Bautin[4].

propositions may be obtained by direct consideration of the stability problem. We shall here consider one problem, which will be very useful in what follows.

Let the root of the characteristic equation $\lambda_1, \ldots, \lambda_n$ have simple elementary divisors. To each root λ_s there then corresponds only a single solution and a single canonical variable $z_s (s = 1, \ldots, n)$. Then the total derivative with regard to t of the expression

$$U^{(\alpha_1 \ldots \alpha_n)} = z_1^{\alpha_1} \ldots z_n^{\alpha_n},$$

where $\alpha_1, \ldots, \alpha_n$ are non-negative integers, has by reason of the first-approximation canonical equations (12) the following form:

$$\frac{dU^{(\alpha_1 \ldots \alpha_n)}}{dt} = (\alpha_1\lambda_1 + \ldots + \alpha_n\lambda_n) U^{(\alpha_1 \ldots \alpha_n)}.$$

We shall consider the first-order partial differential equation

$$\frac{dV}{dt} = W, \tag{13}$$

in which the continuous bounded function W, which vanishes when all x_s are zero, is expanded (in the region of variation of the variables $x_1, \ldots, x_n$ under consideration) in a uniformly convergent series in polynomials $U^{(\alpha_1 \ldots \alpha_n)}$:

$$W = \sum A_{(\alpha_1 \ldots \alpha_n)} U^{(\alpha_1 \ldots \alpha_n)}.$$

We shall seek a solution in the form

$$V = \sum C_{(\alpha_1 \ldots \alpha_n)} U^{(\alpha_1 \ldots \alpha_n)}.$$

Substituting this expression in equation (13) and equating coefficients of $U^{(\alpha_1 \ldots \alpha_n)}$, we obtain:

$$(\alpha_1\lambda_1 + \ldots + \alpha_n\lambda_n) C_{(\alpha_1 \ldots \alpha_n)} = A_{(\alpha_1 \ldots \alpha_n)}.$$

The values $C_{(\alpha_1 \ldots \alpha_n)}$ are defined uniquely if

$$\alpha_1\lambda_1 + \ldots + \alpha_n\lambda_n \gtrless 0.$$

Consequently, if the roots of the characteristic equation $\Delta(\lambda)$ permit n groups of solutions and the expression $\alpha_1\lambda_1 + \ldots + \alpha_n\lambda_n$ never vanishes for non-negative integers α_s, then equation (13) always has a unique formal solution.

If the roots λ_s in the plane of the complex variable are denoted by the points forming them and the latter are loaded by non-negative masses α_s, the expression $\alpha_1\lambda_1 + \ldots + \alpha_n\lambda_n$ will be equal to $M(\xi + i\eta)$ where $M = \alpha_1 + \ldots + \alpha_n$ is the total mass, while ξ, η are the co-ordinates of the centre of gravity of the loaded points. The centre of gravity never passes beyond the limits of the convex region containing the loaded points; therefore, if all roots λ_s lie on one side of a certain straight line, passing through the origin, the expression $\alpha_1\lambda_1 + \ldots + \alpha_n\lambda_n$ can never vanish for any non-negative α_s.

For example, if the real parts of all roots $\lambda_1, \ldots, \lambda_n$ are negative, the equality $\alpha_1\lambda_1 + \ldots + \alpha_n\lambda_n = 0$ cannot exist for non-negative α_s; equation (13) has in this case a unique solution.

If there exist non-negative integers $\beta_1, \ldots, \beta_n$, for which

$$\beta_1\lambda_1 + \ldots + \beta_n\lambda_n = 0,$$

then equation (13) has a solution to within a term

$$C_{(\beta_1 \ldots \beta_n)} U^{(\beta_1 \cdots \beta_n)}$$

with arbitrary coefficient $C_{(\beta_1 \ldots \beta_n)}$, if $A_{(\beta_1 \ldots \beta_n)} = 0$. If $A_{(\beta_1 \ldots \beta_n)} \gtrless 0$, a bounded solution of required form for V does not exist, in the function V

$$[A_{(\beta_1 \ldots \beta_n)} t + C_{(\beta_1 \ldots \beta_n)}] U^{(\beta_1 \cdots \beta_n)}$$

with arbitrary constant $C_{(\beta_1 \ldots \beta_n)}$ corresponds to the term

$$A_{(\beta_1 \ldots \beta_n)} U^{(\beta_1 \cdots \beta_n)}.$$

Consider now the general case. Let

$$\lambda_1, \ldots, \lambda_k$$

be roots of the characteristic equation under the assumption that each root is repeated as many times as the corresponding number of groups of solutions. The canonical variables $z_j^{(s)}$ satisfy equation (12).

Consider the m-th-degree form

$$U_r = \Pi (z_j^{(s)})^{\alpha_j^{(s)}} \qquad \left(\sum_{js} \alpha_j^{(s)} = m\right).$$

There will be a finite number of such forms. These forms

will be lettered so that the product of powers of the variables of all groups (s) $(s = 1, \ldots, k)$ with exponents

$$\alpha_1^{(s)}, \ldots, \alpha_{j-1}^{(s)}, \alpha_j^{(s)}, \ldots, \alpha_{n_s}^{(s)}$$

relates to a number r greater than the number corresponding to the exponents

$$\alpha_1^{(s)}, \ldots, \alpha_{j-1}^{(s)} + 1, \alpha_j^{(s)} - 1, \ldots, \alpha_{n_s}^{(s)}.$$

This may be attained in various ways. Assuming the numbering U_r to have been carried out, we consider the equation

$$\frac{dV}{dt} = \sum_r A_r U_r,$$

where A_r are certain constants. Seeking a solution of this equation in the form

$$V = \sum_r C_r U_r$$

and substituting this expression V in the equation in which we are interested, we find, in accordance with the numbering U_r,

$$\sum_r C_r \left[(\alpha_1 \lambda_1 + \ldots + \alpha_k \lambda_k) U_r + \sum_\mu \beta_\mu^{(r)} U_\mu \right] = \sum_r A_r U_r,$$

where $\alpha_1, \ldots, \alpha_k$ are certain non-negative numbers;

$$\alpha_1 + \ldots + \alpha_k = m; \quad \beta_\mu^{(r)}$$

are certain non-positive numbers when $\mu < r$, and zero when $\mu \geqslant r$. Equating coefficients for identical U_r gives the following algebraic equations for determining the coefficients

$$C_r (\alpha_1 \lambda_1 + \ldots + \alpha_k \lambda_k) + \sum_p \beta_r^{(p)} C_p = A_r.$$

This system of equations will have a non-vanishing solution for C_r, if the determinant D, formed of its coefficients, is non-vanishing. But the determinant D is triangular since in its i-th line all elements up to the i-th will be zero; along the diagonal of the determinant D will be the magnitudes

$$\alpha_1 \lambda_1 + \ldots + \alpha_k \lambda_k.$$

Therefore the determinant D will be non-vanishing if none of the expressions $\alpha_1 \lambda_1 + \ldots + \alpha_k \lambda_k$ vanish for non-negative

integers α_s, whose sum is m. In this case the equation in which we are interested will have for V a single solution.

Paraphrasing this result in the initial variables x_s, we have Lyapunov's theorem: when the roots $\lambda_1, \ldots, \lambda_n$ of the characteristic equation $\Delta(\lambda)$ are such that with given whole positive m no relationships are possible for them in the form

$$\alpha_1\lambda_1 + \ldots + \alpha_n\lambda_n = 0,$$

in which all α_s would be non-negative integers, whose sum is m, then it is always possible to find one, and only one, whole homogeneous function V of degree m in the quantities x_s, satisfying the equation

$$\sum_s (p_{s1}x_1 + \ldots + p_{sn}x_n)\frac{\partial V}{\partial x_s} = U$$

for an arbitrarily given whole homogeneous function U of the quantities x_s in the same degree m.

35. It may be noted that the necessary and sufficient condition for the characteristic polynomial $\Delta(\lambda)$ to have all roots with negative real parts is the positive-definiteness of the quadratic form V, which is a solution of the equation

$$\sum_s (p_{s1}x_1 + \ldots + p_{sn}x_n)\frac{\partial V}{\partial x_s} = U,$$

where U is some arbitrary positive-definite quadratic form of the variables x_s.

Essentially, the sufficiency of this condition follows directly from Lyapunov's theorem on stability and the complement to it on asymptotic stability.

The necessity is found quite simply. If all roots λ_s have negative real parts, then there are no non-negative integers α whose sum is 2 for which there exists a relationship of the form

$$\alpha_1\lambda_1 + \ldots + \alpha_n\lambda_n = 0;$$

and, from the preceding theorem in section 34, this signifies that the equation of interest will have a single solution in some quadratic form V of the variables x_s. Then the form V will be such that it is not possible to make

it negative by choice of values of the variables x_s, since otherwise it would satisfy all conditions of Lyapunov's theorem on instability; but the motion cannot be unstable if all roots of the characteristic equation have negative real parts. Thus V must necessarily be at least positive. If it is assumed that it is reduced to a sum of squares $V = v_1^2 + \ldots + v_m^2$, where v_j denote the independent linear forms in the variables x_s, we note that m cannot be less than n, since the derivative V', vanishing for $v_1 = 0, \ldots, v_m = 0$, could not coincide with the sign-definite function U if $m > n$.

This hypothesis permits us to establish a number of criteria equivalent to Hurwitz's criterion, depending on the form of the negative-definite form U. These criteria consist of inequalities which must hold if all principal diagonal minors of the discriminant of the quadratic form V are to be positive. Although they are in no way simpler than Hurwitz's inequalities, they have the advantage that they permit us to write down the necessary function V by a direct method after the calculations to determine the inequalities.

36. Example. Let us consider a solid body of mass m with moments of inertia A, A, C with regard to the axes x, y, z of the central ellipsoid of inertia. If u, v, w and p, q, r are the projections on these axes of the velocity of the centre of inertia and the angular velocity of the rotating body respectively, the kinetic energy will be:

$$2T = m(u^2 + v^2 + w^2) + (Ap^2 + Aq^2 + Cr^2).$$

Consider that the following forces act on the body: forces

$$(-\sigma u, -\sigma v, -\sigma w),$$

applied at a constant point of the body $(0, 0, l)$, a force $(0, 0, Z)$, applied to the centre of gravity, and a pair of forces with moment $(0, 0, N)$. For simplicity, let the positive functions σ and Z depend on u, v, w, p, q while the function N depends on u, v, w, p, q, r; let $A > C$.

For convenience let us take the differential equations of motion of this body in the well-known Kirchhoff form form:

$$m\frac{du}{dt} = m(rv - qw) - \sigma u, \qquad A\frac{dp}{dt} = (A - C)qr + \sigma lv,$$
$$m\frac{dv}{dt} = m(pw - ru) - \sigma v, \qquad A\frac{dq}{dt} = (C - A)rp - \sigma lu,$$
$$m\frac{dw}{dt} = m(qu - pv) - \sigma w + Z, \qquad C\frac{dr}{dt} = N.$$

Assume that these equations have the particular solution

$$u = 0,\ v = 0,\ w = w_0 > 0,\ p = 0,\ q = 0,\ r = r_0 > 0;$$

this solution exists if

$$Z_0 - \sigma_0 w_0 = 0 \text{ and } N_0 = 0.$$

We want to determine the conditions in which this particular solution will be asymptotically stable in the first approximation with regard to the variables u, v, w, p, q, r.

The differential equations of the first approximation for perturbed motion with regard to the quantities u, v, w, p, q, r **have the following form, if the variations of the variables are denoted respectively by** $\alpha, \beta, \gamma; \xi, \eta, \zeta$:

$$m\frac{d\alpha}{dt} = m(r_0\beta - w_0\gamma) - \sigma_0\alpha,$$
$$m\frac{d\beta}{dt} = m(w_0\xi - r_0\alpha) - \sigma_0\beta,$$
$$m\frac{d\gamma}{dt} = \frac{\partial(Z - \sigma w)}{\partial u}\alpha + \frac{\partial(Z - \sigma w)}{\partial v}\beta +$$
$$+ \frac{\partial(Z - \sigma w)}{\partial w}\gamma + \frac{\partial(Z - \sigma w)}{\partial p}\xi + \frac{\partial(Z - \sigma w)}{\partial q}\eta,$$
$$A\frac{d\xi}{dt} = (A - C)r_0\eta + \sigma_0 l\beta,$$
$$A\frac{d\eta}{dt} = (C - A)r_0\xi - \sigma_0 l\alpha,$$
$$C\frac{d\zeta}{dt} = \frac{\partial N}{\partial u}\alpha + \frac{\partial N}{\partial v}\beta + \frac{\partial N}{\partial w}\gamma + \frac{\partial N}{\partial p}\xi + \frac{\partial N}{\partial q}\eta + \frac{\partial N}{\partial r}\zeta.$$

where all partial derivatives are taken for the values

$$u = v = p = q = 0, \quad w = w_0, \quad r = r_0.$$

For these variational equations the characteristic equation has the form

$$\Delta(\lambda)=\left(C\lambda-\frac{\partial N}{\partial r}\right)\left(m\lambda-\frac{\partial(Z-\sigma v)}{\partial w}\right)\times$$

$$\times\begin{vmatrix} -\sigma_0-m\lambda & mr_0 & 0 & -mw_0 \\ -mr_0 & -\sigma_0-m\lambda & mw_0 & 0 \\ 0 & \sigma_0 l & -A\lambda & (A-C)r_0 \\ -\sigma_0 l & 0 & (C-A)r_0 & -A\lambda \end{vmatrix}=0.$$

From this there follow directly two conditions

$$\frac{\partial N}{\partial r}<0 \text{ and } \frac{\partial(Z-\sigma w)}{\partial w}<0$$

for two obvious roots of the equation $\Delta(\lambda)=0$ to be negative. The condition for the real parts of the other four roots of the equation to be negative may be obtained from Hurwitz's theorem by expanding the fourth-order determinant in a polynomial and calculating the diagonal minors Δ_j of the corresponding Hurwitz matrix. After some calculations, it may be established that Δ_1 and Δ_4 are always positive, the condition $\Delta_3>0$ is equivalent to the inequality $l<0$, with Δ_2 positive. In other words, all real parts of the remaining four roots will be negative if $l<0$.

This result may be obtained much more simply. The fourth-order determinant, the roots of which are of interest, is the characteristic for the first four variational equations. If $l<0$, in the present case ($w_0>0$, $\sigma_0>0$) the function

$$2V=m(\alpha^2+\beta^2)-\frac{mw_0}{\sigma_0 l}A(\xi^2+\eta^2)$$

will be positive-definite with respect to α, β, ξ, η. Its derivative with respect to t

$$V'=-\sigma_0(\alpha^2+\beta^2).$$

is negative.

Consequently, if α or β are distinct from zero, the function V will decrease for change of variables in accordance with the variational equations; if at some moment $\alpha=0$, $\beta=0$, by reason of the first equations, at least in the following sufficiently small time interval α and β will be distinct from zero, if ξ, η are not zero; in other words, V will decrease from any arbitrarily small value for $l<0$. And this proves the result we require.

Example. The question of the choice of parameters of

a stable mechanical system has practical significance.

To have comparable data, we shall consider various states of a single mechanical system described by definite variables.

In the equations of perturbed motion

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n \qquad (s = 1, \ldots, n) \tag{14}$$

let the coefficients p_{sr} be constant and depend on certain parameters. We shall assume that for the values of the parameters considered the unperturbed motion is asymptotically stable.

The positive-definite Lyapunov function V is defined by the equation

$$\sum \frac{\partial V}{\partial x_s}(p_{s1}x_1 + \ldots + p_{sn}x_n) = -(x_1^2 + \ldots + x_n^2). \tag{15}$$

Let

$$2V = \sum a_{sr}x_s x_r \qquad (a_{rs} = a_{sr}).$$

The extreme value of the function V on the sphere $x_1^2 + \ldots + x_n^2 = c$ (c is a positive constant) is found by Lagrange's method by considering

$$V - \lambda(x_1^2 + \ldots + x_n^2 - c).$$

From this

$$\sum (a_{sr} - 2\lambda\delta_{sr})x_r = 0 \qquad (s = 1, \ldots, n); \tag{16}$$

consequently, Lagrange's factor λ should be a root of the secular equation

$$\|a_{sr} - 2\lambda\delta_{sr}\| = 0.$$

Since V is a positive-definite quadratic form all roots λ_s of this equation will be positive; let

$$0 < \lambda_1 \leqslant \ldots \leqslant \lambda_n.$$

Multiplying equalities (16) by x_s and adding, we find that the extreme values of V^* on the sphere

$$x_1^2 + \ldots + x_n^2 = c$$

satisfy the equalities

$$V^* = \lambda c$$

and, consequently,

$$\lambda_1 (x_1^2 + \ldots + x_n^2) \leqslant V \leqslant \lambda_n (x_1^2 + \ldots + x_n^2). \tag{17}$$

From this, by reason of equations (15) and the equations of perturbed motion (14), we have:

$$-\lambda_1 \frac{dV}{dt} \leqslant V \leqslant -\lambda_n \frac{dV}{dt}$$

or

$$V_0 e^{-\frac{t}{\lambda_1}} \leqslant V \leqslant V_0 e^{-\frac{t}{\lambda_n}}. \tag{18}$$

From this we must conclude that the upper bound of possible values of V will be smaller for that system for which λ_n is a minimum if the initial values are the same. If the initial values x_s lie on the sphere

$$x_1^2 + \ldots + x_n^2 = A,$$

from inequality (17) we find

$$\lambda_1 A \leqslant V_0 \leqslant \lambda_n A.$$

As a result of inequalities (18),

$$A\lambda_1 e^{-\frac{t}{\lambda_1}} \leqslant V \leqslant A\lambda_n e^{-\frac{t}{\lambda_n}}.$$

The points $(x_1, \ldots, x_n)$, at which Lyapunov's function has the value V, in accordance with (17), are located either inside or on the sphere

$$x_1^2 + \ldots + x_n^2 = \varepsilon,$$

where

$$\varepsilon = \frac{V}{\lambda_1} \leqslant V_0 \frac{1}{\lambda_1} e^{-\frac{t}{\lambda_n}},$$

and, consequently, one value V_0 at the fixed point in space $(x_1, \ldots, x_n)$ does not characterise the transient period.

For initial perturbations lying on the sphere

$$x_1^2 + \ldots + x_n^2 = A,$$

we have

$$\varepsilon \leqslant A \frac{\lambda_n}{\lambda_1} e^{-\frac{t}{\lambda_n}}.$$

Thus the time for the point $(x_1, \ldots, x_n)$ to pass from the sphere A to the sphere ε is not greater than

$$\lambda_n \ln \frac{A\lambda_n}{\varepsilon\lambda_1}.$$

This limit will be a minimum for given A and ε, if the parameters of the system are taken in accordance with the condition that

$$\lambda_n \ln \frac{A\lambda_n}{\varepsilon\lambda_1}$$

be a minimum.

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CHAPTER 5

THE EFFECT OF PERTURBATION FORCES ON EQUILIBRIUM

Normal Co-ordinates

37. Let us consider a holonomic system in an equilibrium position, for which the values of all its Lagrangian co-ordinates $q_1, \ldots, q_n$ are assumed equal to zero.

The variational equations for the perturbed motion may be obtained if, in the expressions for the kinetic energy T and the potential function V, we retain only terms of the lowest order and, near to the given equilibrium position, take these expressions to be real (symmetrical) quadratic forms with constant coefficients

$$2T = \sum_{ij} a_{ij} q_i' q_j' \quad \text{и} \quad 2V = \sum_{ij} b_{ij} q_i q_j,$$

where $a_{ij} = a_{ji}$ and $b_{ij} = b_{ji}$.

If the system is displaced from the equilibrium position and left to itself, the first-approximation differential equations of perturbed motion will have the form of Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) = -\frac{\partial V}{\partial q_i}$$

or

$$\sum_j a_{ij} q_j'' + \sum_j b_{ij} q_j = 0.$$

These equations may be simplified as follows. Let us pass from co-ordinates $q_1, \ldots, q_n$ to new independent variables $x, \ldots$ by means of linear transformations with constant coefficients

$$q_i = m_i x + \ldots \qquad (i = 1, \ldots, n).$$

The Lagrangian form with respect to the variable x (in the new variables) is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial x'}\right) + \frac{\partial V}{\partial x} = \sum m_i \left[\frac{d}{dt}\left(\frac{\partial T}{\partial q_i'}\right) + \frac{\partial V}{\partial q_i}\right].$$

From this, to obtain Lagrange's equation for the variable x we multiply the equations of motion by constant factors m_i and add:

$$\left(\sum_{ij} m_i a_{ij} q_j\right)'' + \sum_{ij} m_i b_{ij} q_j = 0;$$

we also define the factors m_i such that the relationship

$$x \equiv \sum_{ij} m_i a_{ij} q_j = \frac{1}{\lambda} \sum_{ij} m_i b_{ij} q_j,$$

is valid for all values of the variables q_j; from this

$$\sum_i (\lambda a_{ij} - b_{ij}) m_i = 0. \tag{19}$$

Then Lagrange's equation for x will be very simple:

$$x'' + \lambda x = 0. \tag{20}$$

Equation (19) will have a non-trivial solution for the factors m_i only when λ is a root of the equation

$$\Delta(\lambda) = \| \lambda a_{ij} - b_{ij} \| = 0.$$

By Sylvester's theorem, proved in this case in a similar manner to that in section 18, all the roots of $\Delta(\lambda) = 0$ will be real.

The matrix $\Delta(\lambda)$ has simple elementary divisors. The proof of this proposition, first noted by Weierstrass, may be presented as follows[1]. Let λ be some root of the equation $\Delta(\lambda) = 0$. Without paying attention to the simplicity of the elementary divisors of the system of equations (19) connected with λ, there will be at least one solution for m_i (if the latter is considered to within an arbitrary factor common to all m_i). But this shows that in each case there will exist at least one variable x linearly connected with the

Lagrangian variables $q_1, \ldots, q_n$ and satisfying equation (20). Let us substitute any one of the variables q_s by the variable x; for simplicity of notation we shall assume that we have substituted q_n. After this substitution the expressions for the kinetic energy T and the potential function V must necessarily take the forms

$$2T = x'^2 + \sum_{i,j=1}^{n-1} a_{ij}^* q_i' q_j', \quad 2V = \lambda x^2 + \sum_{i,j=1}^{n-1} b_{ij}^* q_i q_j,$$

as Lagrange's equation, taken with regard to the variable x, must coincide with equation (20). The kinetic energy T in turn is always a positive-definite function with respect to the velocities; therefore the quadratic form

$$\sum_{i,j=1}^{n-1} a_{ij}^* q_i' q_j'$$

is positive-definite with respect to

$$q_1', \ldots, q_{n-1}'.$$

Consequently Lagrange's equations in the variables $q_1, \ldots, q_{n-1}$, will depend only on the sums in the above expressions for T and V and will admit at least one transformation of variables similar to that considered. Continuing this process further and noting that in the transformation of the variables of the kinetic energy, due to its sign-definiteness, there will always remain a sum containing the independent q_s', we can bring T and V in the result to the forms

$$2T = x_1'^2 + \ldots + x_n'^2, \quad 2V = \lambda_1 x_1^2 + \ldots + \lambda_n x_n^2.$$

But this also proves that equations (19) have n non-trivial solutions for $m_1, \ldots, m_n$, i.e. all elementary divisors of the matrix $\Delta(\lambda)$ have a degree equal to unity. The factors m_i can be determined in a similar manner to that described for the A_{ij} in section 24.

The variables x_ν are called normal co-ordinates. Lagrange's equations in normal co-ordinates have the form

$$x_\nu'' + \lambda_\nu x_\nu = 0 \qquad (\nu = 1, \ldots, n).$$

These equations are easily integrated:

$$x_\nu = A_\nu \cos(\sqrt{\lambda_\nu}\, t + B_\nu) \quad \text{for} \quad \lambda_\nu > 0,$$
$$x_\nu = A_\nu t + B_\nu \quad \text{for} \quad \lambda_\nu = 0,$$
$$x_\nu = A_\nu e^{\sqrt{-\lambda_\nu}\, t} + B_\nu e^{-\sqrt{-\lambda_\nu}\, t} \quad \text{for} \quad \lambda_\nu < 0,$$

where A_ν, B_ν denote the constants of integration. From this we conclude that if all the roots λ_ν of the equation $\Delta(\lambda) = \|\lambda a_{ij} - b_{ij}\| = 0$ are positive, the equilibrium is stable in the first approximation and the perturbed motions are harmonic oscillations of the normal co-ordinates x_ν with frequencies $\sqrt{\lambda_\nu}$. For other cases the equilibrium will be unstable in the first approximation and the normal co-ordinate x_s corresponding to a non-positive root λ_s will vary with time, either linearly or exponentially.

It should be emphasized that, in the reduction to normal co-ordinates, an essential role was played by the sign-positiveness of the kinetic energy T. Thus, if in a mechanical system there are some cyclic co-ordinates, although the equations of motion may be brought to the form of Lagrange's equations by ignoring the cyclic co-ordinates (section 9), nevertheless for these equations the function R will not always be positive-definite with regard to the velocities of the non-cyclical co-ordinates and, consequently, in stability problems of steady-state motion it may occur that the normal co-ordinates do not exist.

The Effect of a New Constraint

38. Let us assume that a new constraint is placed on a dynamical system, in the equilibrium position under consideration.

To the first approximation, for small absolute values of the normal co-ordinates x_ν, the new constraint may be given by the equation

$$\sum_\nu A_\nu x_\nu = 0 \quad \text{or} \quad \sum_\nu A_\nu \delta x_\nu = 0,$$

where the constants A_ν do not all vanish, while δx_ν signifies possible variations of the normal co-ordinates. The equations of motion may be obtained from d'Alembert's principle, expressed in normal co-ordinates by the equation

$$\sum_\nu (x_\nu'' + \lambda_\nu x_\nu)\, \delta x_\nu = 0.$$

Multiplying the equation of constraint by the undetermined factor μ and adding to the preceding expression, we have:

$$\sum_{\nu}(x_{\nu}''+\lambda_{\nu}x_{\nu}+\mu A_{\nu})\delta x_{\nu}=0,$$

from which

$$x_{\nu}''+\lambda_{\nu}x_{\nu}+\mu A_{\nu}=0 \qquad (\nu=1, \ldots, n).$$

For those normal co-ordinates for which the constants A_{ν} are equal to zero, the differential equations of motion will retain their previous form and, consequently, the new constraint has no effect on the form of variation of these variables. Only those variables are affected by application of the new constraint which correspond to constants A_{ν} different from zero.

To find the characteristic exponents for the variables x_{ν}, subject to the constraint, we put

$$x_{\nu}=B_{\nu}e^{\sqrt{-1}\,\lambda t}, \quad \mu=Me^{\sqrt{-1}\,\lambda t}$$

and substitute these values both in the preceding equation

$$B_{\nu}(\lambda_{\nu}-\lambda)+A_{\nu}M=0,$$

and in the equation of constraint

$$\sum_{\nu}A_{\nu}B_{\nu}=0.$$

Eliminating B_{ν} from the last relationship, we have

$$f(\lambda)\equiv\sum_{\nu}\frac{A_{\nu}^2}{\lambda-\lambda_{\nu}}=0.$$

Only those $\lambda_{\nu_1}, \ldots, \lambda_{\nu_s}$ of λ_{ν} enter into this expression which correspond to non-zero constants A_{ν}. Considering the signs of the function $f(\lambda)$ in the interval $(\lambda_{\nu_i}, \lambda_{\nu_{i+1}})$ for values of λ close to the ends of the interval (the roots λ_{ν_i} are numbered in order of increasing magnitude), we note that, as λ varies over the interval $(\lambda_{\nu_i}, \lambda_{\nu_{i+1}})$, $f(\lambda)$ will pass from positive values to negative and, consequently, it will have at least one root in this interval. From this we conclude that the characteristic exponents λ for the system subject to the new constraint will alternate with the roots $\lambda_{\nu_1}, \ldots, \lambda_{\nu_s}$. For the variables x_{ν} not subject to the constraint, the numbers λ will coincide with the corresponding values of λ_{ν}.

If $s = 1$ (which occurs when only a single constant A_{ν_1} in the equation of constraint is different from zero) then all values of λ_ν will be possible roots except λ_{ν_1}. The value λ_{ν_1} will appear to have dropped out.

Consequently, if the initial position of equilibrium was stable (all $\lambda_\nu > 0$), the values of λ alternating with λ_ν will also be all positive, and thus the stability of the equilibrium position will be preserved on the application of a constraint.

If the equilibrium position had a degree of instability (the number of non-positive λ_ν) greater than unity, the instability of equilibrium is preserved with application of one new constraint. If the number of non-positive λ_ν was equal to unity, by the application of one appropriate new constraint the equilibrium of the system may be stabilised.

The Effect of Dissipative Forces

39. Let f be a positive-definite real (symmetrical) quadratic form of the velocities x'_ν with constant coefficients $c_{\alpha\beta} = c_{\beta\alpha}$:

$$2f = \sum_{\alpha\beta} c_{\alpha\beta} x'_\alpha x'_\beta.$$

The generalised forces X_ν, the components of which are defined by the relationships

$$X_\nu = -\frac{\partial f}{\partial x'_\nu},$$

have been called dissipative forces by Lord Kelvin. These forces include the ordinary frictional forces, acting at points of a system in the direction opposite to their velocity and proportional to the magnitude of the latter. The function f is termed the scattering function or Rayleigh dissipation function. If this Rayleigh function f contains derivatives of all normal co-ordinates, the dissipation is termed complete, in the contrary case, partial.

Let us assume that, apart from the potential forces, dissipative forces with the Rayleigh function f act on the system.

The equations of small motion of such a system close to its equilibrium position will be:

$$\frac{dx_\nu}{dt} = x'_\nu, \quad \frac{dx'_\nu}{dt} = -\lambda_\nu x_\nu - \frac{\partial f}{\partial x'_\nu}. \tag{21}$$

The effect of dissipative forces on the stability of equilibrium was studied by Lord Kelvin; he proved the following theorems.

Dissipative forces do not disturb stability. In essence, if the equilibrium position is stable under the action of purely potential forces, the λ_ν are all positive.

The total energy of the system

$$H = \frac{1}{2}\sum_\nu x'^2_\nu + \frac{1}{2}\sum_\nu \lambda_\nu x^2_\nu$$

will be a positive-definite quadratic form of the velocities x'_ν and the co-ordinates x_ν. Their total derivative with regard to time is, from equations (21),

$$H' = -2f$$

and H' is a negative-definite function. In accordance with Lyapunov's theorem on stability, we conclude that the equilibrium position remains stable with the addition of dissipative forces.

If the equilibrium is stable with potential forces, it becomes asymptotically stable with the addition of dissipative forces with complete dissipation. Essentially, we consider the function

$$W = H + \beta\sum_\nu x_\nu x'_\nu.$$

The constant β may always be taken so that the function W is positive-definite. The discriminant of W has the form

$$\left\| \begin{array}{cc} \|\delta_{ij}\| & \|\beta\delta_{ij}\| \\ \|\beta\delta_{ij}\| & \|\lambda_j\ \delta_{ij}\| \end{array} \right\|.$$

If all λ_j are positive numbers, it is always possible to take β so small that all principal diagonal minors of this discriminant are positive.

The total derivative of W with respect to time is, from equations (21),

$$W' = -\left[\sum_{ij}(c_{ij} - \delta_{ij}\beta)\,x_i' x_j' + \beta\sum \lambda_i x_i^2 + \beta\sum_{ij} c_{ij} x_i' x_j\right].$$

All principal diagonal minors of the discriminant in square brackets of the form

$$\begin{vmatrix} \| c_{ij} - \delta_{ij}\beta \| & \left\| \frac{\beta}{2} c_{ij} \right\| \\ \left\| \frac{\beta}{2} c_{ij} \right\| & \| \beta\lambda_j \delta_{ij} \| \end{vmatrix}$$

will be positive for sufficiently small positive β (when in determining the sign of the principal diagonal minors it is possible to neglect terms in higher powers of β in comparison with terms in lower powers of β), since the Rayleigh dissipation function f is positive-definite.

Thus, for sufficiently small positive β the function W will be positive-definite; as it is not explicitly dependent on t, it admits an infinitely small upper bound; its derivative W' for this choice of β will be a negative-definite function. The conditions of Lyapunov's theorem on asymptotic stability are satisfied and the proposition is proved.

It is to be understood that, if the dissipation is not complete, the stability of equilibrium existing for only potential forces will not become asymptotically stable by the addition of such dissipative forces.

An isolated equilibrium which is unstable with potential forces cannot be stabilised by dissipative forces.

Proof. Among the λ_ν let there exist at least one negative coefficient and none equal to zero. Consider the function

$$W = H + \beta\sum \lambda_\nu x\, x_\nu'.$$

Its total derivative with respect to time is, by reason of equations (21),

$$W' = -\left[\sum_{\mu\nu}(c_{\mu\nu} - \delta_{\mu\nu}\beta\lambda_\nu)\,x_\mu' x_\nu' + \beta\sum_{\nu}\lambda_\nu^2 x_\nu^2 + \beta\sum \lambda_\nu c_{\mu\nu} x_\mu' x_\nu\right].$$

The discriminant of the quadratic form standing in square brackets is

$$\left\| \begin{array}{c|c} \| c_{\mu\nu} - \delta_{\mu\nu}\beta\lambda_\nu \| & \left\| \frac{\beta}{2} c_{\mu\nu}\lambda_\nu \right\| \\ \hline \left\| \frac{\beta}{2} c_{\mu\nu}\lambda_\nu \right\| & \| \beta\lambda_\nu^2 \delta_{\mu\nu} \| \end{array} \right\|.$$

For sufficiently small positive β (when, in determining the signs of the principal diagonal minors of this discriminant, it is possible to neglect terms in higher powers of β in comparison with terms in lower powers of β), all principal minors will be positive if the dissipation is complete and if none of the λ_ν vanishes. For this choice of β the derivative W' will be a negative-definite function of the variables x'_ν, x_ν. The function W admits an infinitely small upper bound as it is a function not dependent explicitly on t; then by suitable choice of values of x'_ν, x_ν the function W may be made negative in an arbitrarily small neighbourhood near the point $x'_\nu = 0$, $x_\nu = 0$. Thus the function W satisfies all conditions of Lyapunov's theorem on instability, and hence the proposition is proved.

The case of a non-isolated equilibrium position is possible when certain of the λ_ν are zero. If the non-zero roots are all positive, instability in the presence of potential forces exists due to the secular terms appearing in the expressions for the variables x_ν corresponding to the zero roots; such instability may be stabilised by dissipative forces; for example, the force $X_1 = -kx'_1 (k > 0)$ stabilises equilibrium in the case where $\lambda_1 = 0$ and the remaining λ_ν are positive.

The proof of instability with dissipative forces, for the case where among the λ_ν there is at least one negative root and several equal to zero, reduces, with transformation to new variables, to the previously considered

$$x_\nu = y_\nu e^{\varepsilon t},$$

where ε is a suitably chosen constant.

Influence of Gyroscopic Forces

40. The forces

$$X_\nu = \sum_\mu g_{\nu\mu} x'_\mu \quad (g_{\nu\mu} = -g_{\mu\nu}),$$

which do no work in a virtual displacement, have been called gyroscopic by Lord Kelvin.

An equilibrium which is stable under purely potential forces remains stable with the addition of gyroscopic and dissipative forces. With the addition of such forces the differential equations of perturbed motion close to the equilibrium position have the form

$$\frac{dx_\nu}{dt} = x'_\nu,$$
$$\frac{dx'_\nu}{dt} = -\lambda_\nu x_\nu - \frac{\partial f}{\partial x'_\nu} + X_\nu.$$

If the equilibrium is stable for purely potential forces, then all λ_ν are positive. Then the total energy of the system

$$H = \frac{1}{2}\left(\sum_\nu x'^2_\nu + \sum_\nu \lambda_\nu x^2_\nu\right)$$

constitutes a positive definite function. Its total derivative H' by reason of the equations of perturbed motion

$$H' = -2f$$

will not be positive, independently of whether the dissipative forces have total or partial dissipation or none at all ($f = 0$). From this, in accordance with Lyapunov's stability theorem, we obtain immediately Kelvin's proposition on the preservation of stable equilibrium with the addition of gyroscopic and dissipative perturbing forces.

An isolated equilibrium, which is unstable under the action of potential forces, remains unstable with the addition of gyroscopic forces and dissipative forces if the latter have complete dissipation. The proof is easily carried out by consideration of the function

$$W = H + \beta \sum_\nu \lambda_\nu x'_\nu x_\nu,$$

which for non-zero λ and for total dissipation will have a negative-definite derivative W' if the positive constant β is taken sufficiently small. The function W, not depending explicitly on t, admits an infinitely small upper bound; for arbitrarily small absolute magnitudes of x'_ν, x_ν they may be taken such that W is negative, as among the λ_ν will exist at least one that is negative. The instability follows from Lyapunov's theorem of section 14.

If the instability of an isolated position of equilibrium

under the action of exclusively potential forces has an odd degree, gyroscopic stabilisation of the equilibrium is not possible. In essence, in the absence of dissipative forces (which is not a necessary condition and adopted only to simplify the formulae) the equations of motion will be

$$x_\nu'' = -\lambda_\nu x_\nu + \sum_\mu g_{\nu\mu} x_\mu' \qquad (g_{\mu\nu} = -g_{\nu\mu}).$$

Let there be an odd number of negative λ_ν. For an isolated equilibrium position all λ_ν are non-vanishing. The characteristic equation for this system of equations of perturbed motion is

$$\Delta(\lambda) = \begin{vmatrix} \lambda_1 + \lambda^2 & \dots & -g_{1n}\lambda \\ \dots & \dots & \dots \\ -g_{n1}\lambda & \dots & \lambda_n + \lambda^2 \end{vmatrix} = 0.$$

We have:

$$\Delta(0) = \lambda_1 \dots \lambda_n \text{ and } \Delta(\lambda)_{\lambda\to\infty} > 0.$$

For an odd number of negative λ_ν the expressions $\Delta(0)$ and $\Delta(\infty)$ will have opposing signs. Consequently, the equation $\Delta(\lambda) = 0$ will have at least one positive root for λ. This proves the instability.

For certain conditions of equilibrium, which are unstable under the action of purely potential forces, it is possible to improve or stabilise the system by the addition of suitable gyroscopic forces if the degree of instability is not odd and if the added dissipative forces do not have complete dissipation.

To prove the possibility of gyroscopic stabilisation in such cases, let us define pairs of normal co-ordinates corresponding to negative λ_ν. Let one such pair of variables have the following equations with addition of gyroscopic forces

$$x'' = -\alpha x + gy', \quad y'' = -\beta y - gx',$$

where α and β are negative. The characteristic equation of such a system

$$\Delta(\lambda) = \lambda^4 + \lambda^2(g^2 + \alpha + \beta) + \alpha\beta = 0$$

will have purely imaginary roots if it has negative roots

for λ^2. The latter occurs if

$$g^2+\alpha+\beta>0,\ (g^2+\alpha+\beta)^2-4\alpha\beta>0.$$

If the constant g satisfies the last inequality, the equilibrium will be stabilised. The proposition is proved.

But if on a dynamical system forces act with dissipation of energy for any real displacements, gyroscopic stabilisation is not possible as proved above. And since in reality small dissipative forces with complete dissipation always exist, the gyroscopic stabilisation has in such circumstances a temporary significance and is violated as soon as dissipative forces with complete dissipation act on the system. Therefore Lord Kelvin proposed calling an equilibrium obtained by gyroscopic stabilisation temporary stability and equilibrium stability existing under the action of purely potential forces he proposed calling secular.

Example. The centre of gravity of an artillery shell moves in the vertical plane of fire $\xi\zeta$ along the ξ-axis, which for simplicity we take horizontal. Let x be the axis of shell; v the constant velocity of its centre of gravity; l the projection of the axis of the shell on the plane of fire; α the angle between l and x; β the angle between ξ and l; ξ the horizontal axis, ζ the vertical and η the third axis of the left-hand system $(\xi\eta\zeta)$. This problem is a familiar approximation for the motion of a shell in a very flat trajectory.

The moving system of axis (xyz), having the origin at the centre of gravity of the shell, is defined kinematically, to establish the holonomicity of the variables which define the position of the shell with regard to its centre of gravity. Initially we rotate the system $(\xi\eta\zeta)$ about the η-axis by the angle β such that the ξ-axis passes into the l-axis, the new position of the ζ-axis is termed z, and we obtain the system $(l\eta z)$; this is rotated about the z-axis by the angle α such that the l-axis pass into the x-axis, and the system obtained is termed (xyz). Let ω denote the angle of rotation of the shell about its x-axis in the system (xyz). From the kinematic definition, the holonomicity of the variables ω, α, β follows directly.

The instantaneous velocity of rotation of the shell is the resultant of angular velocities ω', α', β', along the positive x-, z-, η-axis respectively; from this the projections p, q, r

of the angular velocity on the x-, y-, z-axis will be equal to

$$p = \omega' + \beta' \sin \alpha, \quad q = \beta' \cos \alpha, \quad r = \alpha'.$$

The kinetic energy of rotary motion of the shell has the well-known form

$$2T = Ap^2 + Bq^2 + Cr^2,$$

where A, B, C denote the moments of inertia of the shell with regard to the x-, y-, z-axis. For ordinary elongated shells, which are bodies of revolution about their axis, all these moments of inertia are constant magnitudes and $C = B > A$.

The moment of the turning couple acting on the shell is directed orthogonally to the plane passing through the velocity of the centre of gravity v and the axis of the shell x on the side from which rotation from v to x is positive. Other couples acting on the shell are not taken into consideration. The magnitude of this couple $K = a \sin \gamma$ depends on the positive magnitude $a = a(v)$ and on the angle γ between the velocity of flight v and the x-axis; $\cos \gamma = \cos \alpha \cos \beta$. The magnitudes of the generalised Lagrangian forces Q are obtained directly from the expression for the virtual work of the couple

$$a \sin \gamma \, \delta\gamma = -a\delta \cos \gamma = a \sin \alpha \cos \beta \delta\alpha + a \cos \alpha \sin \beta \delta\beta.$$

The differential equations of rotary motion of the shell are Lagrange's equations for the variables ω, α, β. One of the equations leads directly to the first integral $Ap = \text{const}$, corresponding to the cyclical co-ordinate ω; the two others have the form established by Krylov:

$$B\alpha'' + B\beta'^2 \sin \alpha \cos \alpha - Ap\beta' \cos \alpha = a \sin \alpha \cos \beta,$$
$$B\beta'' \cos \alpha - 2B\alpha'\beta' \sin \alpha + Ap\alpha' = a \sin \beta.$$

Multiplying these equations once by α', $\beta' \cos \alpha$, and then by $\sin \beta$, $\sin \alpha \cos \beta$ respectively, and adding individually each time, after some simple transformations and integration we obtain the two first integrals:

$$\frac{B}{2}(\alpha'^2 + \beta'^2 \cos^2 \alpha) + a \cos \alpha \cos \beta = h,$$
$$B(\alpha' \sin \beta - \beta' \sin \alpha \cos \alpha \cos \beta) + Ap \cos \alpha \cos \beta = k.$$

For the unperturbed motion of the shell we take the

particular solution of the differential equations of motion given by $\alpha = 0$, $\beta = 0$, $\alpha' = 0$, $\beta' = 0$. For this unperturbed motion the equations of perturbed motion will be Krylov's differential equations. The first of these integrals is the integral $H = h$, obtained by the method (section 9) of ignoring the cyclic co-ordinate ω; here the function $V = H - H_0$ is not sign-definite.

To solve the stability problem with respect to the variables α, β, α', β' we multiply the first of the integrals by Ap, the second by $-2a$ and add:

$$W = \frac{BAp}{2}(\alpha'^2 + \beta'^2 \cos^2 \alpha) - \\ - 2Ba(\alpha' \sin \beta - \beta' \sin \alpha \cos \alpha \cos \beta) - Apa \cos \alpha \cos \beta.$$

The first approximation integral $W + Apa = W_2 + \dots$ is

$$W_2 = \frac{BAp}{2}\alpha'^2 - 2Ba\alpha'\beta + \frac{Apa}{2}\beta^2 + \\ + \frac{BAp}{2}\beta'^2 + 2Ba\beta'\alpha + \frac{Apa}{2}\alpha^2.$$

The quadratic form W_2 consists of two quadratic forms of the same type; it is sufficient to consider one of them

$$f = \frac{BAp}{2}\alpha'^2 - 2Ba\alpha'\beta + \frac{Apa}{2}\beta^2.$$

The form f will then be sign-definite with respect to the variables entering into it if its discriminant

$$\begin{vmatrix} \frac{BAp}{2} & -Ba \\ -Ba & \frac{Apa}{2} \end{vmatrix},$$

is positive, i.e.

$$A^2p^2 - 4Ba > 0.$$

This is the required condition. If it is satisfied, then the form f will be sign-definite with regard to α', β: together with the form f, the form W_2 will be sign-definite with regard to α, β, α', β' and thus also the integral $W + Apa$. Thus $W + Apa$ satisfies all the conditions of Lyapunov's theorem on stability; from this we conclude that if the last inequality is observed, the unperturbed motion of the shell under study ($\alpha = \beta = \alpha' = \beta' = 0$) will be stable.

The stabilisation of the shell has a gyroscopic nature and in accordance with Kelvin's proposition should be lost with the existence of complete dissipation. Let us add to the first approximation of the differential equations of motion small dissipative forces, derivatives of some Rayleigh function

$$2f = b\alpha'^2 + 2e\alpha'\beta' + c\beta'^2,$$

where the small constant coefficients b, e, c satisfy the condition of positive-definiteness $b > 0$, $bc - e^2 > 0$. We will have

$$B\alpha'' = a\alpha + Ap\beta' - b\alpha' - e\beta',$$
$$B\beta'' = \alpha\beta - Ap\alpha' - e\alpha' - c\beta'.$$

For a completely rigorous proof that the dissipative forces disrupt the gyroscopic stabilisation, let us consider the quadratic form

$$W = \frac{1}{2}B(\alpha'^2 + \beta'^2) - \frac{1}{2}(\alpha^2 + \beta^2) - \varepsilon B(\alpha\alpha' + \beta\beta').$$

The positive constant ε will be defined later. The derivative of W, by reason of the first-approximation equation, will be:

$$W' = -[(b + 2\varepsilon B)\alpha'^2 + 2e\alpha'\beta' - 2\varepsilon b\alpha'\alpha + 2(Ap - e)\varepsilon\alpha'\beta + (c + 2\varepsilon B)\beta'^2 - 2(Ap + e)\varepsilon\beta'\alpha - 2c\varepsilon\beta'\beta + 2\varepsilon a\alpha^2 + 2\varepsilon a\beta^2].$$

The discriminant of the quadratic form $-W'$ is

$$\begin{vmatrix} b + 2\varepsilon B & e & -b\varepsilon & (Ap - e)\varepsilon \\ e & c + 2\varepsilon B & -(Ap + e)\varepsilon & -c\varepsilon \\ -b\varepsilon & -(Ap + e)\varepsilon & 2a\varepsilon & 0 \\ (Ap - e)\varepsilon & -c\varepsilon & 0 & 2a\varepsilon \end{vmatrix}.$$

Removing from the third and fourth columns of this determinant the common positive factor ε, we conclude that for sufficiently small ε (when, in determining the signs of all principal diagonal minors of the discriminant, it is possible to neglect terms in higher powers of ε in comparison with terms in lower powers) these minors will be positive; consequently, for such ε the given form W' will be negative-definite.

The form W admits an infinitely small upper bound, being

a function not dependent explicitly on time; by choosing numerically arbitrarily small α, β, α', β', it is always possible to make the form W negative, i.e. of the same sign as W'. From this, by reason of Lyapunov's theorem on instability, we conclude that the dissipative forces violate the gyroscopic stabilisation of the shell and that gyroscopic stabilisation of the shell is temporary.

Certain Forced Motions

41. The majority of forced motions of some mechanical system originate basically in the motion of some other system which acts on the first and, in turn, is subject to reaction by the first. Frequently the motion of the second system is considered as given approximately by neglecting the reaction of the first system[2].

Let us consider the case where both systems are in perturbed motion close to the equilibrium position; we shall carry out calculations in normal co-ordinates for the simplest possible case. Let the motion of the two systems (x) and (y) be given by the equations

$$\begin{cases} x'' = -\alpha x - \varkappa x' + \mu y', \\ y'' = -\beta y. \end{cases} \tag{22}$$

This case may occur if in the system (y) the existing gyroscopic forces $-\mu x'$ are neglected; the dissipation is partial. The constants $\varkappa$ and μ are positive.

We shall consider the case where α and β are simultaneously positive, in other words, the case where if the gyroscopic force $-\mu x'$ were not neglected there would be secular stability in Kelvin's sense. For simplicity let $\alpha = n^2$, $\beta = p^2$ and

$$y = \frac{A}{\mu p} \sin pt.$$

The equation for the system (x) takes the form

$$x'' + \varkappa x' + n^2 x = A \cos pt.$$

Let us take

$$x = a \cos (pt - \varepsilon)$$

then, substituting in the equation, we have

$$a(n^2 - p^2)\cos(pt - \varepsilon) - \varkappa pa \sin(pt - \varepsilon) =$$
$$= A\cos\varepsilon\cos(pt - \varepsilon) - A\sin\varepsilon\sin(pt - \varepsilon),$$

from which, equating coefficients of cos $(pt-\varepsilon)$ and sin $(pt-\varepsilon)$, we have:

$$a(n^2 - p^2) = A\cos\varepsilon, \qquad ap\varkappa = A\sin\varepsilon,$$

so that the (particular) solution may be written in the following manner:

$$x = \frac{A\sin\varepsilon}{p\varkappa}\cos(pt - \varepsilon),$$

where

$$\tan\varepsilon = \frac{p\varkappa}{n^2 - p^2}.$$

Such an oscillation is called a forced oscillation; it is the response to the force $\mu y'$, acting from without on the system (x). The amplitude of forced oscillation is proportional to the amplitude of the force A and the period is the same as the period of the force.

The work of the perturbing force on the forced motion of the system (x) during its period is always positive

$$\mu\int y'x'\,dt = \frac{A^2\sin^2\varepsilon}{p\varkappa}\int_0^{2\pi}\cos^2 pt\,dpt -$$
$$- \frac{A^2\sin\varepsilon\cos\varepsilon}{p\varkappa}\int_0^{2\pi}\sin pt\cos pt\,dpt = \frac{\pi A^2\sin 2\varepsilon}{p\varkappa},$$

since

$$\int_0^{2\pi}\cos^2 u\,du = \pi, \qquad \int_0^{2\pi}\sin u\cos u\,du = 0.$$

The work done has a maximum magnitude for $\sin^2\varepsilon = 1$, i.e. when the phase of forced oscillation lags behind the phase of the force by a quarter-period, which occurs if the period of the perturbing force coincides with the natural period of oscillation of the system (x), in other words $n = p$.

In the case of equal periods, friction should be taken into consideration, no matter how small it is and how negligible

its result for $n \neq p$.

This consideration was expressed by Rayleigh.

If friction is absent and the periods are equal, $n = p$, then the corresponding characteristic λ-matrix

$$\begin{Vmatrix} \lambda^2 + n^2 & -\mu\lambda \\ 0 & \lambda^2 + n^2 \end{Vmatrix}$$

has a multiple elementary divisor

$$(\lambda^2 + n^2)^2,$$

which signifies that, no matter how small the amplitude of oscillation y, secular terms necessarily appear. The equilibrium of the system (x) will be unstable. The variational equations for the forced motion

$$y = \frac{A}{\mu p} \cos pt, \qquad x = \frac{A}{2p} t \sin pt$$

will be of the same type except that in place of x, y their variations will appear; the elementary divisors will be the same, which means that the forced motion will be unstable.

When the periods are equal, $n = p$, and there is friction $\varkappa > 0$, the elementary divisors for the characteristic λ-matrix of the system (22)

$$\begin{Vmatrix} \lambda^2 + \varkappa\lambda + n^2 & -\mu\lambda \\ 0 & \lambda^2 + n^2 \end{Vmatrix}$$

will be simple

$$(\lambda^2 + n^2), \quad \lambda^2 + \varkappa\lambda + n^2.$$

Thus for arbitrarily small friction secular terms are now absent from the solution of system (22): the equilibrium of the system will be stable in the sense that it is always possible to find a small number A such that the absolute values of x do not exceed a pre-assigned bound.

When the period of the force and the period of natural oscillations of the system (x) do not agree, from the expression

$$d \frac{\tan \theta}{d\varkappa} = \frac{p}{n^2 - p^2}$$

an interesting conclusion follows. When the period of the force is greater than the period of natural oscillations of the system (x), the effect of increased friction results in the appearance of a phase delay which varies from zero to a quarter-period. If, on the contrary, the period of natural oscillation x is the greater, the initial delay of a half-period decreases to somewhat less than a quarter-period; in other words, the effect of increased friction consists in a phase shift between the force and the forced oscillation tending to a quarter-period of the force.

42. In the present chapter some very significant propositions of Lord Kelvin have been established. The perturbed motions of a mechanical system close to the stable position of equilibrium, where the potential function of the forces acting has a minimum, have an oscillatory character and vice versa. The addition of dissipative forces does not alter the stability or instability of equilibrium of a dynamical system, which exists under potential forces. The addition of gyroscopic forces without disturbing the stability of such positions of equilibrium, may, in certain cases, where the equilibrium has an instability of even degree, while the dissipative forces do not have complete dissipation, stabilise unstable equilibrium.

The first of these propositions has its rigorous expression in Lagrange's theorem on the stability of an isolated position of equilibrium at the maximum of the force function. Concerning the other propositions, the question arises in connection with them whether the phenomena noted by Lord Kelvin from consideration of the first approximation of the equations of perturbed motion will or will not retain validity in reality.

Many serious engineering problems lead to this last problem.

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CHAPTER 6

FIRST-APPROXIMATION STABILITY

Fundamental Theorem

Previous to the work of Lyapunov, investigations of stability were limited to consideration of the first approximation only. It is possible to come to erroneous conclusions in such studies. Lyapunov posed and solved the question as to when the equations of first approximation fully determine the problem of stability and instability.

43[24L]. Let the differential equations of perturbed motion have the form

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n + X_s \qquad (s = 1, \ldots, n), \tag{23}$$

where X_s are holomorphic functions of $x_1, \ldots, x_n$, the expansion beginning with terms of not lower than second order, while p_{sr} are constants.

Lyapunov's theorem. If the real parts of all roots $\lambda_s = \alpha_s + i\beta_s$ of the first-approximation characteristic equation are negative, the unperturbed motion is asymptotically stable, independently of infinitesimal terms of higher order than the first in X_s.

Proof. If the real parts of all roots λ_s are negative, then for any non-negative $m_1, \ldots, m_n$, whose sum is 2, the expression

$$m_1\lambda_1 + \ldots + m_n\lambda_n.$$

cannot vanish. Consequently, there exists a negative-definite quadratic form with positive coefficients W satisfying the equation

$$\sum \frac{\partial W}{\partial x_s}(p_{s1}x_1 + \ldots + p_{sn}x_n) = x_1^2 + \ldots + x_n^2.$$

Then

$$\frac{dW}{dt} = x_1^2 + \ldots + x_n^2 + \sum \frac{\partial W}{\partial x_s} X_s$$

will be positive-definite independently of X_s, which proves the theorem.

44[24L]. Theorem. If among the roots λ_s of the characteristic equation is found at least one with positive real part, the unperturbed motion is unstable, independently of infinitesimal terms of higher order than the first.

Proof. Let the root λ_1 have the greatest positive real part and let $\varkappa$ be a positive number, smaller than $\alpha_1 = \operatorname{Re}\lambda_1$.

The equation

$$\sum \frac{\partial V}{\partial x_s}(p_{s1}x_1 + \ldots + p_{sn}x_n) = 2\varkappa V + x_1^2 + \ldots + x_n^2$$

in accordance with Euler's theorem on homogeneous functions

$$2V = \sum \frac{\partial V}{\partial x_s} x_s$$

may be written in the following manner:

$$\sum \frac{\partial V}{\partial x_s}[p_{s1}x_1 + \ldots + (p_{ss} - \varkappa)x_s + \ldots + p_{sn}x_n] = \\ = x_1^2 + \ldots + x_n^2.$$

This equation is of a known type; the characteristic equation, related to it

$$\Delta(\mu) = \|p_{sr} - \delta_{sr}(\varkappa + \mu)\| = 0$$

will have as roots the quantities

$$\mu_s = \lambda_s - \varkappa,$$

of which at least μ_1 will have positive real part. Whatever the roots λ_s are, it is always possible to find $\varkappa < \operatorname{Re}\lambda_1$ such that, for any non-negative $m_1, \ldots, m_n$ whose sum is 2, the expression

$$m_1\lambda_1 + \ldots + m_n\lambda_n - 2\varkappa.$$

will not vanish. But this means that the function V exists and, by suitable choice of variables $x_1, \ldots, x_n$, may be

made positive.

From this

$$\frac{dV}{dt} = 2\varkappa V + \left(x_1^2 + \ldots + x_n^2 + \sum \frac{\partial V}{\partial x_s} X_s\right).$$

In the parenthesis, there is a positive-definite function, independent of X_s. According to Lyapunov's theorem instability follows directly.

Note. The preceding theorem of Lyapunov is not applicable, of course, only to the case where X_s does not depend explicitly on t, as the latter circumstance did not play any particular role in the proof. The coefficients $P_s^{(m_1 \cdots m_n)}$ in the functions X_s may be continuous, bounded, real functions of t, satisfying the inequalities

$$|P_s^{(m_1 \cdots m_n)}| < \frac{M}{A^{m_1+\ldots+m_n}},$$

where A is some positive constant.

45. Example. In the first approximation the differential equations of perturbed motion of a servo-mechanism have the form

$$\begin{aligned} T_a\xi' - \eta - \zeta &= 0, \\ T_r^2\eta'' + T_k\eta' + \delta\eta + \xi &= 0, \\ T_s\zeta' - \eta &= 0, \end{aligned}$$

where all coefficients in the variations of the variables of the machine ξ, the regulator η and the servo-motor ζ are positive constants, depending in a known manner on the design elements of the system.

The question is posed as to what conditions must be satisfied by these conditions for stability with perturbed initial values of the variables ξ, η, η', ζ independently of the further terms of second and higher orders in the exact equations of perturbed motion. Such questions arise whenever the pre-assigned precision does not allow strict confidence in terms of higher than the first order.

In accordance with the theorems proved, for stability in the first approximation to exist, the characteristic equation

$$T_s T_a T_r^2 \lambda^4 + T_a T_k T_s \lambda^3 + \delta T_a T_s \lambda^2 + T_s \lambda + 1 = 0$$

should have all roots with negative real parts, for which Hurwitz's theorems give the inequalities[1]

$$\delta T_k T_a > T_r^2 \quad \text{and} \quad T_s(\delta T_k T_a - T_r^2) > T_a T_k^2.$$

Example. Gyroscopic stabilization of an equilibrium position which is unstable under purely conservative forces has a temporary character, while the disturbance of gyroscopic stabilization due to the action of dissipative forces with complete dissipation cannot be changed by terms of higher than first order in the equations of perturbed motion, since the first approximation admits a Lyapunov function with sign-definite derivative having a quadratic form (section 40). Therefore gyroscopic stabilization of rotary motion of a projectile is temporary as long as dissipative forces with complete dissipation do not act on the system.

Critical Cases

46[1]. The theorems proved leave unclarified cases where certain of the roots of the characteristic equations have negative real parts, while other roots have these parts equal to zero. These cases are critical in the sense that their stability or instability cannot be determined by consideration of only the first approximation.

In these critical cases further approximations of X_s may give either stability or instability. In other words, the functions X_s may always be taken so that either stability or instability occurs, according to desire.

Without loss of generality it is possible to limit consideration to a single group of equations corresponding to a multiple elementary divisor of the root, the real part of which is equal to zero.

Consider first the zero root:

$$\frac{dz_1}{dt} = Z_1,$$

$$\frac{dz_i}{dt} = -z_{i-1} + Z_i \qquad (i = 2, \ldots, m).$$

Let us consider the functions $\varphi_1, \ldots, \varphi_m$, defined by the recursive equation

$$\varphi_s = z_s^2 + \varphi_{s+1}^2 \qquad (s = m,\ m-1,\ \ldots,\ 1)$$

under the condition $\varphi_{m+1} = 0$.

If we take

$$Z_s = 2z_{s+1}\varphi_{s+1} - z_s f_s,$$

then the function φ_1, positive-definite by definition, will have as its total derivative with regard to time the function

$$\varphi_1' = -2z_1^2 f_1 - 2^2\varphi_2 z_2^2 f_2 - \\ - 2^3\varphi_2\varphi_3 z_3^2 f_3 - \ldots - 2^m\varphi_2 \ldots \varphi_m z_m^2 f_m,$$

which is negative-definite or positive-definite depending on whether all f_s have been taken positive-definite or if they have all been taken negative-definite. In the first case the unperturbed motion will be asymptotically stable, in the second unstable.

For pairs of purely imaginary roots $\lambda = \pm i\beta$ let us consider the group

$$\frac{dz_1}{dt} = i\beta z_1 + Z_1,$$
$$\frac{dz_j}{dt} = i\beta z_j - z_{j-1} + Z_j \quad (j = 2,\ \ldots,\ m).$$

Let the functions $\varphi_1, \ldots, \varphi_m$ be defined successively by the equations

$$\varphi_s = z_s\bar{z}_s + \varphi_{s+1}^2$$

under the condition

$$\varphi_{m+1} = 0.$$

If we take

$$Z_s = 2z_{s+1}\varphi_{s+1} - z_s f_s,$$

where f_s have sign-definite real parts, the positive-definite function φ_1 will have as its total derivative with regard to time

$$\varphi_1' = -z_1\bar{z}_1(f_1 + \bar{f}_1) - 2\varphi_2 z_2\bar{z}_2(f_2 + \bar{f}_2) - \ldots - \\ - 2^{m-1}\varphi_2 \ldots \varphi_m z_m\bar{z}_m(f_m + \bar{f}_m)$$

which is positive-definite or negative-definite depending on whether the functions $\mathrm{Re} f_s$ are all negative-definite or all positive-definite. The unperturbed motion will in the first case be unstable and in the second case asymptotically stable. Thus, terms of higher orders may reduce the instability of the first approximation which exists because of secular terms in the solution of the variational equations.

If the functions X_s are constrained by some structural limitations, the question of stability in certain cases is solved in the first approximation without regard to the presence of the critical case. For example, if the equations of perturbed motion have the form of canonical Hamiltonian equations, the equilibrium will be stable if in the Hamiltonian function the lowest quadratic terms constitute a sign-definite function.

47. The theorems proved on stability in the first approximation permit direct solution also of the question of stability in the presence of perturbing forces if the latter add in the differential equations of the perturbed motion (23) infinitesimal terms of not less than second order. This is very important practically, since the most frequently encountered stability problems in engineering presuppose the existence not only of deviation from the initial values of the variables but the existence of certain small perturbing forces, not completely defined.

We shall confine ourselves in the proof assumed for the first-approximation stability theorem (section 43) to those properties only of the functions X_s which were used in it. Let us introduce the same function here. If all roots of the characteristic polynomial $\Delta(\lambda)$ have negative real parts, the equation

$$\sum (p_{s1}x_1 + \ldots + p_{sn}x_n)\frac{\partial V}{\partial x_s} = x_1^2 + \ldots + x_n^2$$

has a solution in the form of a negative-definite quadratic form V (sections 34, 35). We have

$$V' = x_1^2 + \ldots + x_n^2 + \sum_s \frac{\partial V}{\partial x_s} X_s.$$

Let us turn now to the proof of the theorem of section 8 on stability, and particularly to Lyapunov's method proposed for deriving the numbers λ for a given number A. We shall

define also those constraints which it is sufficient to place on the function X_s, so that the derivation of the number λ for the given sufficiently small number A does not depend on the numerical values of X_s.

Our function V does not depend on t and is negative-definite. Let l be the exact lower bound of its absolute value on the sphere (A)

$$\sum x_s^2 = A.$$

For positive l there is found, from the property of the possible infinitely small upper bound of the function V (Section 7), such a non-zero number λ that for

$$\sum_{s} x_s^2 \leqslant \lambda$$

the inequality

$$|V| < l.$$

will be satisfied. Let us take the initial perturbations x_{s0} in accordance with the inequality

$$\sum x_s^2 \leqslant \lambda.$$

From the relationship

$$V - V_0 = \int_{t_0}^{t} V' \, dt \tag{24}$$

we deduce that, if in the region

$$\sum x_s^2 \leqslant A$$

the functions X_s are constrained by the inequalities

$$|X_s| < \lambda, \tag{25}$$

then

$$|V| \leqslant |V_0| < l. \tag{26}$$

In fact, for A so small that under the condition

$$\sum x_s^2 \leqslant A$$

the inequality

$$1-\sum_s\left|\frac{\partial V}{\partial x_s}\right|>0,$$

will be satisfied under the assumptions (25) on the functions X_s, we will have for values of the variables x_s satisfying the conditions

$$\lambda\leqslant\sum x_s^2\leqslant A,$$

independently of the numerical values of X_s,

$$V'=x_1^2+\ldots+x_n^2+\sum\frac{\partial V}{\partial x_s}X_s\geqslant$$
$$\geqslant\sum x_j^2\left(1-\frac{\sum\left|\frac{\partial V}{\partial x_s}\right||X_s|}{\lambda}\right)>0$$

From the last inequality and the relationship (19) there directly follows inequality (21) and thus the following:

$$\sum x_j^2<A,$$

since l is the exact lower bound of the absolute value of V on the sphere (A).

If for an arbitrarily small positive number A the function X_s may be subject to the inequalities (25), where λ signifies a number constructed by Lyapunov's method, the unperturbed motion will be stable independently of the numerical values of X_s.

In this analysis the method should be noted, for a given positive number A, less than H, satisfying the condition

$$1-\sum\left|\frac{\partial V}{\partial x_s}\right|>0 \text{ for } \sum x_s^2\leqslant A,$$

of finding with the aid of the function V a positive number λ having the property: if the initial values of x_{s0} and the functions X_s are subject to the inequalities

$$\sum x_{s0}^2\leqslant\lambda,\quad |X_s|<\lambda\quad (s=1,\ldots,n),$$

then for all succeeding time the values of the variables x_s will satisfy the inequality

$$\sum_{s} x_s^2 < A.$$

We shall not develop this analysis further*

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* Questions of stability under constantly acting perturbations were considered by Duboshin[3], Artem'ev[4] and Malkin[5].

CHAPTER 7

THE CASE OF A SINGLE ZERO ROOT

Auxiliary Transformation

48[28L]. Let the characteristic equation have a single root, equal to zero, the other roots having negative real parts. In this case the equations of the first approximation will have a single linear integral, as is obvious from their canonical form in section 30. Taking this integral x as a variable, we bring the equations of perturbed motion to the form

$$\frac{dx}{dt} = X,$$
$$\frac{dx_s}{dt} = p_{s1}x_1 + \cdots + p_{sn}x_n + p_s x + X_s \qquad (s = 1, \ldots, n),$$

where X, X_s are holomorphic functions of variables $x, x_1, \ldots, x_n$, beginning with terms of not less than second order; the constants p_{sj} are such that the equation

$$\Delta(\lambda) = \|p_{sj} - \delta_{sj}\lambda\| = 0$$

has all its roots with negative real parts.

Let us denote by X^0, X_s^0 the values of the functions X, X_s, when all the variables $x_1, \ldots, x_n$ are equated to zero. By simple transformations it is always possible to make the lowest degree of x in the function X^0 not higher than that for the functions X_s^0, if X^0, X_s^0 do not vanish identically, and such that all constants p_s be zero.

Essentially, we shall consider the following system of the equations

$$p_{s1}x_1 + \cdots + p_{sn}x_n + p_s x + X_s = 0 \qquad (s = 1, \ldots, n).$$

This system of equations is satisfied by zero values of the variables $x, x_1, \ldots, x_n$. Its functional determinant with regard to the variables $x_1, \ldots, x_n$ is non-vanishing for zero

values of $x, x_1, \ldots, x_n$ and equal to $\Delta(0)$. Therefore there exists the solution

$$x_s = u_s(x) \qquad (s = 1, \ldots, n),$$

where u_s are certain holomorphic functions of x, vanishing when the variable x is equal to zero.

If in the initial system p_s and X_s^0 are zeros, the solution u_s will also be zeros. But in this case the system satisfying the given requirements does not require additional transformation.

If this is not the case, we introduce the new variables $z_1, \ldots, z_n$ in accordance with the equalities

$$x_s = z_s + u_s.$$

The transformed system will have the form

$$\begin{cases} \dfrac{dx}{dt} = Z, \\ \dfrac{dz_s}{dt} = p_{s1}z_1 + \ldots + p_{sn}z_n + Z_s \qquad (s = 1, \ldots, n), \end{cases} \tag{27}$$

where Z denotes the function X after substitution of $x_s = z_s + u_s$, while Z_s denotes the expression

$$p_{s1}u_1 + \ldots + p_{sn}u_n + p_s x + X_s - \frac{du_s}{dx} X$$

after the same substitution. Denoting by Z^0, Z_s^0 the values of the functions Z, Z_s, when the variables z_s are all equated to zero, and taking into account the equations defining u_s, we derive the relationship

$$Z_s^0 = -\frac{du_s}{dx} Z^0,$$

from which it follows that in the expansion of Z_s^0 in powers of x there will be no terms of degree lower than the lowest degree in the expansion of Z^0 in powers of x. If Z^0 vanishes identically, the same is true for Z_s^0. With this transformation the stability problem with regard to $x, x_1, \ldots, x_n$ is equivalent to the stability problem with regard to $x, z_1, \ldots, z_n$. Therefore in the analysis of stability it is possible to start from equations (27).

Analysis of Various Cases

49[29L]. We shall first consider cases where the lowest power of x in the expansion of Z^0 is even. First let this be the second power:

$$Z = gx^2 + Px + Q + R,$$

where P is a linear and Q is a quadratic form of the variables $z_1, \ldots, z_n$; R does not have terms of degree lower than the third; g is a constant.

Let us consider the function

$$V = x + Ux + W,$$

where U is a linear and W is a quadratic form of the variables $z_1, \ldots, z_n$. By means of equations (27) we have:

$$\frac{dV}{dt} = gx^2 + Px + Q + R + UZ + \\ + x \sum_s \frac{\partial U}{\partial z_s}(p_{s1}z_1 + \ldots + p_{sn}z_n + Z_s) + \\ + \sum_s \frac{\partial W}{\partial z_s}(p_{s1}z_1 + \ldots + p_{sn}z_n + Z_s).$$

The functions U, W thus far not defined, will be defined such that

$$\sum_s \frac{\partial U}{\partial z_s}(p_{s1}z_1 + \ldots + p_{sn}z_n) + P = 0,$$
$$\sum_s \frac{\partial W}{\partial z_s}(p_{s1}z_1 + \ldots + p_{sn}z_n) + Q = g(z_1^2 + \ldots + z_n^2).$$

From section 34 we conclude that this definition of U, W is always possible since all roots of the polynomial

$$\Delta(\lambda) = \|p_{sr} - \delta_{sr}\lambda\|$$

have negative real parts. For such U and W we have

$$\frac{dV}{dt} = g(x^2 + z_1^2 + \ldots + z_n^2) + S,$$

where the lowest-order terms in S are of not less than third order.

The derivative $\frac{dV}{dt}$ represents a sign-positive function

with sign coinciding with the sign of the constant g; V does not depend explicitly on t and therefore possesses an infinitely small upper bound. The function V may be made to have the same sign as g and this, by reason of Lyapunov's instability theorem (section 14), shows that in this case the unperturbed motion is unstable.

Let us now consider the general case

$$Z = gx^m + P^{(1)}x + \ldots + P^{(m-1)}x^{m-1} + Q + R,$$
$$Z_s = P_s^{(1)}x + \ldots + P_s^{(m-1)}x^{m-1} + Z_s',$$

where $P^{(i)}$, $P_s^{(i)}$ are linear forms and Q is a quadratic form of the variables $z_1, \ldots, z_n$;

$$R = vx^m + \sum_{ij} v_{ij}z_iz_j, \quad Z_s' = v^{(s)}x^m + \sum_{ij} v_{ij}^{(s)}z_iz_j,$$

where v, $v^{(s)}$, v_{ij}, $v_{ij}^{(s)}$ denote holomorphic functions of the variables $x, z_1, \ldots, z_n$, while v, v_{ij} vanish when the latter all become zero.

If m is an even number, we consider the function

$$V = x + U^{(1)}x + \ldots + U^{(m-1)}x^{m-1} + W,$$

where $U^{(i)}$ are linear forms and W is a quadratic form of the variables z_s.

In accordance with equations (27) we have:

$$\frac{dV}{dt} = gx^m + P^{(1)}x + \ldots + P^{(m-1)}x^{m-1} + Q + R + \sum_{r=1}^{m-1} rU^{(r)}x^{r-1}Z +$$
$$+ \sum_{r=1}^{m-1}\sum_s x^r \frac{\partial U^{(r)}}{\partial z_s}\left[p_{s1}z_1 + \ldots + p_{sn}z_n + \sum_{i=1}^{m-1} P_s^{(i)}x^i + Z_n'\right] +$$
$$+ \sum_s \frac{\partial W}{\partial z_s}\left[p_{s1}z_1 + \ldots + p_{sn}z_n + \sum_{i=1}^{m-1} P_s^{(i)}x^i + Z_s'\right].$$

We define the functions $U^{(r)}$, W $(r = 1, \ldots, m-1)$ in accordance with the equations

$$\sum_s \frac{\partial U^{(r)}}{\partial z_s}(p_{s1}z_1+\ldots+p_{sn}z_n)+\sum_{\alpha+\beta=r}\sum_s \frac{\partial U^{(\alpha)}}{\partial z_s}P_s^{(\beta)}+P^{(r)}=0,$$

$$\sum_s \frac{\partial W}{\partial z_s}(p_{s1}z_1+\ldots+p_{sn}z_n)+Q=g(z_1^2+\ldots+z_n^2).$$

For this choice of the functions $U^{(r)}$, W always possible in the present case, we have:

$$\frac{dV}{dt}=g(x^m+z_1^2+\ldots+z_n^2)+S,$$

where

$$S=wx^m+\sum_{ij} w_{ij}z_iz_j,$$

w, w_{ij} denoting holomorphic functions of the variables $x, z_1, \ldots, z_n$, which vanish when the latter all become zero.

The function V, not depending on t, has an infinitely small upper bound and with suitable choice of arbitrarily small absolute magnitude of $x, z_1, \ldots, z_n$ may take on values of either sign. Its derivative $\frac{dV}{dt}$ is sign-definite with sign coinciding with that of the constant g. Consequently, by Lyapunov's theorem (section 13), the unperturbed motion is in this case (m even) unstable.

We now consider m odd.

The function V will be sought in the form

$$V=\frac{1}{2}x^2+U^{(1)}x^2+\ldots+U^{(m-1)}x^m+W,$$

where $U^{(i)}$ are linear forms and W a quadratic form of the variables z_s. We have:

$$\frac{dV}{dt}=gx^{m+1}+P^{(1)}x^2+\ldots+P^{(m-1)}x^m+xQ+xR+$$
$$+\sum_{r=1}^{m-1}(r+1)U^{(r)}x^rZ+\sum_{r=1}^{m-1}\sum_s x^{r+1}\frac{\partial U^{(r)}}{\partial z_s}\Big(p_{s1}z_1+\ldots+p_{sn}z_n+$$
$$+\sum_{i=1}^{m-1}P_s^{(i)}x^i+Z_s'\Big)+\sum_s\frac{\partial W}{\partial z_s}(p_{s1}z_1+\ldots+p_{sn}z_n+Z_s).$$

Let us obtain the quadratic form W and the linear forms $U^{(r)}$ ($r=1, \ldots, m-1$) in accordance with the equations

$$\sum_s \frac{\partial U^{(r)}}{\partial z_s}(p_{s1}z_1+\ldots+p_{sn}z_n)+\sum_{\alpha+\beta=r}\sum_s \frac{\partial U^{(\alpha)}}{\partial z_s}P_s^{(\beta)}+P^{(r)}=0,$$

$$\sum_s \frac{\partial W}{\partial z_s}(p_{s1}z_1+\ldots+p_{sn}z_n)=g(z_1^2+\ldots+z_n^2).$$

This is always possible since all roots of the polynomial

$$\Delta(\lambda)=\| p_{sr}-\delta_{sr}\lambda \|$$

have negative real parts (section 34). For this choice of $U^{(r)}$, W

$$\frac{dV}{dt}=g(x^{m+1}+z_1^2+\ldots+z_n^2)+S,$$

where

$$S=vx^{m+1}+\sum_{ij}v_{ij}z_iz_j,$$

v, v_{ij} denoting certain holomorphic functions of the variables x, $z_1, \ldots, z_n$, which vanish when the latter all become equal to zero.

The derivative $\frac{dV}{dt}$ is a sign-definite function; its sign agrees with the sign of the constant g.

If g is positive, then since by suitable choice of values of the variables x, $z_1, \ldots, z_n$ the function V may be made positive or, in other words, may have the same sign as that of the derivative $\frac{dV}{dt}$, we conclude that the unperturbed motion is unstable.

If g is negative, then since the quadratic form W will (by section 35) be positive-definite with regard to $z_1, \ldots, z_n$, we conclude that the function V will be positive-definite with regard to all variables x, $z_1, \ldots, z_n$ and, not depending explicitly on t, will have an infinitely small upper bound. On the basis of Lyapunov's theorem on asymptotic stability (section 12) we conclude that in this case the unperturbed motion is stable and each sufficiently close perturbed motion tends to it asymptotically.

Thus the unperturbed motion will be unstable if m is an even number and also if m is odd while g is positive; if m is an odd number and the constant g negative, the unperturbed motion is asymptotically stable.

50. Example. Let us consider a problem which in a certain sense may be an approximation to the problem of the stability of the longitudinal motion of rectilinear flight of a neutral aeroplane.

The aeroplane flying in air represents, together with the air, a mechanical system consisting of a solid body and a continuous medium. In practical investigations such a system is usually simplified and only the aeroplane is considered subject to given accelerating forces. We shall assume that these accelerating forces do not depend explicitly on time and on the co-ordinates of the centre of gravity of the aeroplane, at least for all perturbed motion sufficiently close to the given rectilinear flight of the aeroplane in the homogeneous undisturbed atmosphere.

Let m be the mass of the aeroplane; v the velocity of its centre of gravity G; J the central moment of inertia; φ the angle between the velocity vector v and the horizontal plane; ϑ the angle of the chord of the wing to the horizontal plane; $\alpha = \vartheta - \varphi$, the angle of attack. Let the thrust of the propeller $\Phi(v, \omega)$ pass through the centre of gravity G and have a constant direction with respect to the airplane, making an angle β with the velocity v. The lift force, the resistance force and the resulting moment (with respect to the centre of gravity) will be denoted by $c_a v^2$, $c_w v^2$, and $c_m v^2$. We shall assume that c_m depends neither on v nor on the angle ϑ, when the variables of the problem are taken as v, α, ϑ, $\vartheta' = \omega$. For simplicity we assume that c_a, c_w depend only on α.

The natural equations of longitudinal motion of the aeroplane in its plane of symmetry are

$$m\frac{dv}{dt} = \Phi\cos\beta - mg\sin\varphi - c_w v^2,$$

$$mv\frac{d\varphi}{dt} = \Phi\sin\beta - mg\cos\varphi + c_a v^2,$$

$$J\frac{d^2\vartheta}{dt^2} = c_m v^2.$$

We shall consider the steady-state rectilinear flight in which the variables v, α, ϑ have constant values, which satisfy the equations

$$\Phi\cos\beta - mg\sin\varphi - c_w v^2 = 0,$$

$$\Phi\sin\beta - mg\cos\varphi + c_a v^2 = 0,$$

$$c_m = 0.$$

Denoting by ξ, η, ζ, ζ' the variations of the variables v, α, ϑ, ϑ', respectively, we have the following system of differential equations for these variations (with $\eta = \delta\alpha = \delta\beta$):

$$m\frac{d\xi}{dt} = \xi\left(-2c_w v + \frac{\partial\Phi}{\partial v}\cos\beta\right) - \eta\left(\frac{\partial c_w}{\partial\alpha} - c_a\right)v^2 - \\ - \zeta mg\cos\varphi + \zeta'\frac{\partial\Phi}{\partial\omega}\cos\beta,$$

$$mv\frac{d\eta}{dt} = \xi\left(-2c_a v - \frac{\partial\Phi}{\partial v}\sin\beta\right) - \eta\left(\frac{\partial c_a}{\partial\alpha} + c_w\right)v^2 - \\ - \zeta mg\sin\varphi + \zeta'\left(mv - \frac{\partial\Phi}{\partial\omega}\sin\beta\right),$$

$$\frac{d\zeta}{dt} = \zeta',$$

$$J\frac{d\zeta'}{dt} = \eta\frac{\partial c_m}{\partial\alpha}v^2 + \zeta'\frac{\partial c_m}{\partial\omega}v^2.$$

We shall consider the problem of the stability of steady-state rectilinear flight for a neutral aeroplane, i.e. one which has in the given unperturbed motion the property

$$\frac{\partial c_m}{\partial\alpha} = 0.$$

The characteristic equation will be:

$$\Delta(\lambda) = -\lambda\left(\frac{\partial c_m}{\partial\omega}v^2 - I\lambda\right)\times \\ \times\begin{vmatrix} \frac{\partial\Phi}{\partial v}\cos\beta - 2c_w v - m\lambda & -\left(\frac{\partial c_w}{\partial\alpha} - c_a\right)v^2 \\ -\frac{\partial\Phi}{\partial v}\sin\beta - 2c_a v & -\left(\frac{\partial c_a}{\partial\alpha} + c_w\right)v^2 - mv\lambda \end{vmatrix} = 0.$$

From this we conclude that the characteristic equation for the problem of longitudinal stability of a neutral aeroplane in rectilinear flight with the above assumptions always has a single zero root. Let the other roots have negative real parts; then Hurwitz's inequalities

$$\frac{\partial c_m}{\partial\omega} < 0,$$

$$v\left(\frac{\partial\Phi}{\partial v}\cos\beta - 2c_w v\right) - \left(\frac{\partial c_a}{\partial\alpha} + c_w\right)v^2 < 0,$$

$$\left(\frac{\partial\Phi}{\partial v}\cos\beta - 2c_w v\right)\left(\frac{\partial c_a}{\partial\alpha} + c_w\right) + \left(\frac{\partial\Phi}{\partial v}\sin\beta + 2c_a v\right)\left(\frac{\partial c_w}{\partial\alpha} - c_a\right) < 0.$$

are satisfied.

The question of stability with respect to infinitesimal terms of higher-order should be solved by Lyapunov's

method. It should first be noted that for practical calculation of the constant g, introduced in sections 48, 49, it is not necessary in every case to calculate the functions u_s completely; it is sufficient to determine the terms of degree not higher than $(m-1)$ in x.

In the present problem the linear integral x for the variational equations is

$$x = J\zeta' - \frac{\partial c_m}{\partial \omega} v^2 \zeta.$$

To simplify the calculations, we eliminate the variable ζ and we shall consider the complete system of differential equations of perturbed motion in the variables x,

$$x_1 = \xi, \quad x_2 = \eta, \quad x_3 = \zeta'.$$

We have:

$$\frac{dx}{dt} = X = c_m(\alpha + x_2, \ \omega + x_3)(v + x_1)^2 - \frac{\partial c_m}{\partial \omega} v^2 x_3.$$

We shall consider the expression

$$Z^{(0)} = X(x, \ v + u_1, \ \alpha + u_2, \ \omega + u_3).$$

Equating the right-hand side of the differential equation for x_3 to zero we have

$$c_m(\alpha + u_2, \ \omega + u_3) = 0.$$

For a neutral airplane let all derivatives $\frac{\partial^\mu c_m}{\partial \alpha^\mu}$ vanish up to order $p-1$ inclusive, the derivative $\frac{\partial^p c_m}{\partial \alpha^p}$ being non-zero.

From the last equation we have:

$$u_3 = -\frac{1}{p!} \frac{\frac{\partial^p c_m}{\partial \alpha^p}}{\frac{\partial c_m}{\partial \omega}} u_2^p + \dots$$

Consequently,

$$Z^{(0)} = c_m(\alpha + u_2, \ \omega + u_3)(v + u_1)^2 - \frac{\partial c_m}{\partial \omega} v^2 u_3 = \\ = \frac{v^2}{p!} \frac{\partial^p c_m}{\partial \alpha^p} u_2^p + \dots$$

If $u_2 = Ax^q + \dots$, the degree m of the lowest term gx^m in

the expansion of the function $Z^{(0)}$ in series of increasing powers of x is $m = pq$, while the coefficient

$$g = \frac{v^2}{pl} \frac{\partial^p c_m}{\partial \alpha^p} A^p.$$

For stability of unperturbed motion the degree m should be odd, and the constant g negative; therefore the numbers p and q should both be odd and the inequality $g < 0$ should be satisfied.

The lowest terms of the function u_2 are defined from the system of equations

$$\left(\frac{\partial \Phi}{\partial v} \cos \beta - 2c_w v\right) u_1 - \left(\frac{\partial c_w}{\partial \alpha} - c_a\right) v^2 u_2 + \frac{mg \cos \varphi}{\frac{\partial c_m}{\partial \omega} v^2} x = 0,$$

$$-\left(\frac{\partial \Phi}{\partial v} \sin \beta + 2c_a v\right) u_1 - \left(\frac{\partial c_a}{\partial \alpha} + c_w\right) v^2 u_2 + \frac{mg \sin \varphi}{\frac{\partial c_m}{\partial \omega} v^2} x = 0,$$

from which

$$A = \frac{\left[\left(\frac{\partial \Phi}{\partial v} \cos \beta - 2c_w v\right) \sin \varphi + \left(\frac{\partial \Phi}{\partial v} \sin \beta + 2c_a v\right) \cos \varphi\right] \frac{mg}{\frac{\partial c_m}{\partial \omega} v^4}}{\left(\frac{\partial \Phi}{\partial v} \cos \beta - 2c_m v\right)\left(\frac{\partial c_a}{\partial \alpha} + c_w\right) + \left(\frac{\partial \Phi}{\partial v} \sin \beta + 2c_a v\right)\left(\frac{\partial c_w}{\partial \alpha} - c_a\right)}.$$

When A is positive, as it is for aeroplanes*, if the preceding inequalities are satisfied, the last condition of instability takes the form $\frac{\partial^p c_m}{\partial \alpha^p} < 0$, p odd.

51[30, 31L]. There remains the consideration of the cases where $Z^{(0)}$, $Z_s^{(0)}$ vanish identically. Then equations (27) have the obvious solution

$$x = c, \; z_1 = 0, \; \ldots, \; z_n = 0,$$

depending on the arbitrary constant c.

Let us substitute

$$x = c + z,$$

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* This addition to the treatment in my lectures on the stability of aircraft, making it unnecessary to consider the case $A < 0$, is due to Grossman[1].

understanding by c an arbitrary real constant with absolute magnitude not exceeding some sufficiently small number.

We obtain:

$$Z_s = c_{s1}z_1 + \ldots + c_{sn}z_n + Z'_s \qquad (s = 1, \ldots, n),$$

where c_{sr} are holomorphic functions of the constant c, vanishing for $c = 0$, while Z'_s will be holomorphic functions of the variables $z, z_1, \ldots, z_n$, the expansions of which start with terms of order not lower than the second and have coefficients holomorphic with respect to c. The function

$$Z = c_1z_1 + \ldots + c_nz_n + Z'.$$

is of the same type.

Let us consider a partial differential equation

$$\sum_s \left((p_{s1} + c_{s1})z_1 + \ldots + (p_{sn} + c_{sn})z_n + Z'_s\right)\frac{\partial z}{\partial z_s} = \\ = c_1z_1 + \ldots + c_nz_n + Z'$$

we shall find the holomorphic solution for it in the form

$$z = \sum_{i=1}^{\infty} V_i,$$

where V_i denotes the form of degree i with respect to the variables. To determine the form V_i, comparison of like terms gives the following system of equations:

$$\sum \left[(p_{s1} + c_{s1})z_1 + \ldots + (p_{sn} + c_{sn})z_n\right]\frac{\partial V_i}{\partial z_s} = U_i,$$

where U_i represents some known homogeneous forms of the variables $z_1, \ldots, z_n$ of degree i, depending on the form V_j with index j not exceeding $i-1$. As a result of our assumption that all roots of the equation

$$\Delta(\lambda) = \|p_{ij} - \delta_{ij}\lambda\| = 0$$

have negative real parts, if c is in absolute magnitude sufficiently small all roots of the equation

$$D(\lambda) = \|p_{ij} + c_{ij} - \delta_{ij}\lambda\| = 0$$

will also have negative real parts. For such c the conditions of the theorem in section 34 will be satisfied and,

consequently, the last set of equations uniquely define all V_i in order of increasing index i. All coefficients in the forms V_i will be holomorphic functions of the constant c for sufficiently small absolute values of the latter.

From this we conclude that

$$x = c + z(z_1, \ldots, z_n, c)$$

represents some integral of the system (22) which is improper with regard to the constant c and, consequently, it may replace the first of the differential equations of this system. Then the other equations of the system are brought to the form

$$\frac{dz_s}{dt} = (p_{s1} + c_{s1})z_1 + \ldots + (p_{sn} + c_{sn})z_n + Z'_s, \tag{28}$$

where Z'_s will be holomorphic functions of $z_1, \ldots, z_n$ with coefficients which are holomorphic functions of the constant c; here the expansion of Z'_s will begin with terms of not less than second order in $z_1, \ldots, z_n$.

Our stability problem with respect to the quantities $z, z_1, \ldots, z_n$ is equivalent to the problem of stability with respect to the quantities $c, z_1, \ldots, z_n$. Actually, to all numerically sufficiently small values of one set of magnitudes correspond arbitrarily small numerical values of the others.

For each c sufficiently small in absolute magnitude, the question of stability with respect to the quantities $z_1, \ldots, z_n$ leads to the investigation of equations (28), the characteristic equation of which

$$D(\lambda) = \|p_{ij} + c_{ij} - \delta_{ij}\lambda\| = 0$$

has, for such c, all roots with negative real parts. As a result of this, for each c sufficiently small in absolute magnitude and for each given positive a, there is found such a positive number ε that, if the condition

$$\sum_i z_{i0}^2 < \varepsilon$$

is satisfied at the initial time, then at all succeeding times the following relation is satisfied:

$$\sum_i z_i^2 < a$$

and for these same conditions the functions z_s tend to zero with unlimited increase in t.

However from this we still cannot conclude that the unperturbed motion is stable. For this conclusion to be valid, it is necessary that for $|c|$, not exceeding some number, it is possible to choose ε independently of c.

Let us consider the negative-definite quadratic form V, defined by the equation

$$\sum_s (p_{s1}z_1 + \ldots + p_{sn}z_n)\frac{\partial V}{\partial z_s} = z_1^2 + \ldots + z_n^2.$$

The total derivative with respect to t of this function V is by reason of equation (28),

$$\frac{dV}{dt} = z_1^2 + \ldots + z_n^2 + \sum_s (c_{s1}z_1 + \ldots + c_{sn}z_n + Z_s')\frac{\partial V}{\partial z_s}.$$

The constants c_{ij} are holomorphic functions of c, vanishing for c equal to zero, which means that for values c, sufficiently small in absolute magnitude, the right-hand side of the last equality will be a positive-definite function for sufficiently small absolute values of the variables $z_1, \ldots, z_n$, independently of the value of c (provided it is less than a certain sufficiently small number). This proves the possibility of taking the number ε independently of c (section 8).

Thus, we come to the conclusion that in the case where Z^0, Z_s vanish, the unperturbed motion is stable while a perturbed motion sufficiently close to the unperturbed will approach asymptotically to some steady-state motion

$$x = c,\ z_1 = 0,\ \ldots,\ z_n = 0.$$

52[32L]. Example. There is given the system

$$\frac{dx}{dt} = ax^2 + bxy + cy^2,\quad \frac{dy}{dt} = -y + kx + lx^2 + mxy + ny^2$$

with constants a, b, c, k, l, m, n.

From the equation

$$-y + kx + lx^2 + mxy + ny^2 = 0$$

we find

$$y = u(x) = kx + B_2x^2 + B_3x^3 + \ldots,$$

where

$$B_2 = l + mk + nk^2,$$
$$B_3 = (m + 2nk)\,B_2, \ldots$$

From this

$$Z^0 = ax^2 + bxu + cu^2 = A_2x^2 + A_3x^3 + A_4x^4 + \ldots,$$

where

$$A_2 = a + bk + ck^2,$$
$$A_3 = (b + 2ck)\,B_2,$$
$$A_4 = (b + 2ck)\,B_3 + cB_2^2, \ldots$$

If A_2 is different from zero, the unperturbed motion is unstable.

Let $A_2 = a + bk + ck^2 = 0$. If also $B_2 = 0$ (which implies the vanishing of all remaining coefficients B and A), the unperturbed motion will be stable.

Let us assume that B_2 is not zero. Stability will be determined by the sign of A_3, when $b + 2ck$ is non-zero. If $b + 2ck = 0$, then for c different from zero, the coefficient A_4 will be non-vanishing and, consequently, instability will occur; with $c = 0$, from the equations $b + 2ck = 0$, $A_2 = 0$, it follows that $a = 0$, $b = 0$, and thus $Z^{(0)} = 0$; the unperturbed motion will then be stable.

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CHAPTER 8

THE CASE OF A PAIR OF PURELY IMAGINARY ROOTS

Transformation of the Equations

53[33L]. By determining the real canonical variables (section 30) corresponding to a pair of purely imaginary roots $\pm i\lambda$, the system of differential equations of perturbed motion may be transformed in this case to the following form:

$$\begin{cases} \dfrac{dx}{dt} = -\lambda y + X, \qquad \dfrac{dy}{dt} = \lambda x + Y, \\ \dfrac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n + \alpha_s x + \beta_s y + X_s \\ (s = 1, \ldots, n). \end{cases} \tag{29}$$

Here X, Y, X_s are holomorphic functions of the real variables x, y, $x_1, \ldots, x_n$, the expansion of which begins with terms of not lower than second order and has constant real coefficients; p_{sr}, α_s, β_s are certain real constants, such that all roots of the equation

$$\| p_{sr} - \delta_{sr}\varkappa \| = 0$$

have negative real part; for concreteness we shall consider λ to be a positive constant.

It is possible to assume that the functions X and Y vanish when x and y become zero. Actually, we shall first assume that this does not occur. We define x and y as holomorphic functions of the variables $x_1, \ldots, x_n$, satisfying the equations

$$\sum_s (p_{s1}x_1 + \ldots + p_{sn}x_n + \alpha_s x + \beta_s y + X_s)\frac{\partial x}{\partial x_s} = -\lambda y + X,$$

$$\sum_s (p_{s1}x_1 + \ldots + p_{sn}x_n + \alpha_s x + \beta_s y + X_s)\frac{\partial y}{\partial x_s} = \lambda x + Y$$

and containing in their expansions terms of not lower than second order

$$x = u_2 + u_3 + \dots, \quad y = v_2 + v_3 + \dots,$$

where u_m, v_m are homogeneous functions of degree m in the variables $x_1, \dots, x_n$. For the determination of u_m, v_m we obtain the equations

$$\sum_s (p_{s1}x_1 + \dots + p_{sn}x_n)\frac{\partial u_m}{\partial x_s} = -\lambda v_m + P_m,$$

$$\sum_s (p_{s1}x_1 + \dots + p_{sn}x_n)\frac{\partial v_m}{\partial x_s} = \lambda u_m + Q_m,$$

where P_m, Q_m are homogeneous functions of $x_1, \dots, x_n$ of degree m, which depend on u_k, v_k $(k < m)$.

We shall assume that all the functions u_k, v_k $(k < m)$ have been determined. Then, retaining the notation of section 34, we shall seek a solution for u_m, v_m in the form

$$u_m = \sum A_r U_r, \quad v_m = \sum B_r U_r,$$

where the P_m, Q_m are known functions of $x_1, \dots, x_n$:

$$P_m = \sum p_r U_r, \quad Q_m = \sum q_r U_r.$$

Substituting these values in the previous equation, we obtain for the determination of the constants A_r, B_r the system of linear algebraic non-homogeneous equations

$$A_r(m_1x_1 + \dots + m_nx_n) + \sum_s \beta_r^{(s)} A_s = -\lambda B_r + p_r,$$

$$B_r(m_1x_1 + \dots + m_nx_n) + \sum_s \beta_r^{(s)} B_s = \lambda A_r + q_r,$$

where $m_1, \dots, m_n$ are some non-negative integers, whose sum is m; $\varkappa_1, \dots, \varkappa_n$ are roots of the equations

$$\|p_{ij} - \delta_{ij}\varkappa\| = 0;$$

$\beta_r^{(s)}$ are certain non-positive constants when $r < s$, and zero when $r \geq s$. The determinant of this system of equation is

$$D = \prod [(m_1\varkappa_1 + \dots + m_n\varkappa_n)^2 + \lambda^2].$$

Since, in accordance with the assumption, all roots $\varkappa_s$

have negative real parts, the expression $m_1x_1+\ldots+m_nx_n$ cannot be equal to $\pm\lambda i$ for any non-negative numbers $m_1,\ldots,m_n$, whose sum is m; consequently, the determinant D does not vanish and in the last equations the constants A_r, B_r will be uniquely defined.

Let

$$x=u=u_2+u_3+\ldots, \quad y=v=v_2+v_3+\ldots$$

be the solution found in this way. Introducing the variables $\bar{x}, \bar{y}$ instead of the variables x, y

$$x=u+\bar{x}, \quad y=v+\bar{y},$$

we transform these equations to a form in which the terms $\bar{X}, \bar{Y}$ will vanish when $\bar{x}, \bar{y}$ become zero.

Therefore we assume that in assembling the equations (29) this transformation has been carried out (if necessary) and that, consequently, the functions X and Y vanish when x, y become zero.

54[33L]. Instead of the variables x, y we introduce the polar co-ordinates r, θ:

$$x=r\cos\theta, \quad y=r\sin\theta.$$

We obtain

$$\frac{dr}{dt}=X\cos\theta+Y\sin\theta, \quad r\frac{d\theta}{dt}=\lambda r+Y\cos\theta-X\sin\theta.$$

From the assumption that X, Y vanish when x, y become zero, we conclude that the right-hand sides of the last equations vanish when r becomes zero. Therefore, after cancelling by r, the second equation gives:

$$\frac{d\theta}{dt}=\lambda+\Theta,$$

where Θ denotes a holomorphic function of the variables $r, x_1, \ldots, x_n$ (which vanishes when the latter become simultaneously equal to zero) and having in its expansion coefficients which are rational functions of $\sin\theta$ and $\cos\theta$. From this equation it is evident that as long as the quantities $|r|$, $|x_s|$ do not exceed certain constants, Θ will be a continuous increasing function of t. From this it is clear that in solution of the stability problem the variable θ may play the same role as t. We shall take it as the independent

variable in place of t. Then

$$\left.\begin{aligned} \frac{dr}{d\theta} &= rR, \\ \frac{dx_s}{d\theta} &= q_{s1}x_1 + \ldots + q_{sn}x_n + (a_s \cos\theta + b_s \sin\theta)\, r + Q_s, \end{aligned}\right\} \tag{30}$$

where R, Q_s denote functions of the same character as θ; then the functions Q_s will not contain in their expansions terms of lower than the second order with respect to the quantities r, x_s;

$$q_{sr} = \frac{p_{sr}}{\lambda}, \quad a_s = \frac{\alpha_s}{\lambda}, \quad b_s = \frac{\beta_s}{\lambda}.$$

The first of equations (30) shows that if the initial value of r is zero, then r will be equal to zero for all θ. Consequently r will retain the sign of its initial value at least as long as the quantities r, x_s all remain sufficiently small in absolute magnitude. We shall assume that r does not take on negative values.

55[34, 35L]. For solution of the problem, equations (30) are subjected to a certain transformation, founded on the possibility of their having a periodic solution

$$\left.\begin{aligned} r &= c + u^{(2)}c^2 + u^{(3)}c^3 + \ldots, \\ x_s &= u_s^{(1)}c + u_s^{(2)}c^2 + u_s^{(3)}c^3 + \ldots, \end{aligned}\right\} \tag{31}$$

where c is an arbitrary constant, and $u^{(k)}$, $u_s^{(k)}$ are periodic functions of θ, with common period 2π, independent of c.

This periodic solution will not always be possible.

To determine the functions $u^{(k)}$, $u_s^{(k)}$ we substitute (31) in the system of differential equations (30) and compare coefficients of like powers of c. We obtain:

$$\begin{aligned} \frac{du^{(k)}}{d\theta} &= U^{(k)}, \\ \frac{du_s^{(k)}}{d\theta} &= q_{s1}u_1^{(k)} + \ldots + q_{sn}u_n^{(k)} + (a_s \cos\theta + b_s \sin\theta)\, u^{(k)} + U_s^{(k)}, \end{aligned}$$

where $U^{(k)}$, $U_s^{(k)}$ are known rational functions of those $u^{(\mu)}$, $u_s^{(\mu)}$, for which $\mu < k$, having coefficients which are rational functions of $\sin\theta$ and $\cos\theta$; for $k = 1$ all functions U are zero; $k = 1, 2, \ldots$

The functions $u_s^{(1)}$ will always be periodic of the form

$$u_s^{(1)} = A_s \cos \theta + B_s \sin \theta,$$

where A_s, B_s are certain constants. This may be proved most simply by reduction of the equations for $u_s^{(k)}$ to canonical form (section 30)

$$\frac{dv_s^{(k)}}{d\theta} = \varkappa_s v_s^{(k)} - \sigma_s v_{s-1}^{(k)} + (a_s' \cos \theta + b_s' \sin \theta) u^k + V_s^{(k)},$$

where $v_1^{(k)}, \ldots, v_n^{(k)}$ are canonical variables while the constants a_s', b_s' are linear combinations of the coefficients a_s, b_s; the new $V_s^{(k)}$ will also be known rational functions of those $u^{(\mu)}$, $v_s^{(\mu)}$, for which $\mu < k$, with coefficients which are rational functions of $\sin \theta$ and $\cos \theta$; all $V_s^{(1)}$ are equal to zero; $\varkappa_1, \ldots, \varkappa_n$ are roots of the equation

$$\| q_{ij} - \delta_{ij} \varkappa \| = 0,$$

The constants σ_s are either 0 or 1; $\sigma_1 = 0$, $s = 1, \ldots, n$; $k = 1, 2, \ldots$

From these equations we have:

$$v_s^{(k)} = e^{\varkappa_s \theta} \int_{-\infty}^{\theta} e^{-\varkappa_s \theta} [(a_s' \cos \theta + b_s' \sin \theta) u^{(k)} - \sigma_s v_{s-1}^{(k)} + V_s^{(k)}] d\theta,$$

and, consequently, each time that the expressions shown here in square brackets are whole rational functions of $\sin \theta$ and $\cos \theta$, the functions $v_s^{(k)}$ will be also, and thus $u_s^{(k)}$ as well, being linearly dependent on the latter[1].

For $k = 1$ the functions $u_s^{(1)}$ are obtained in the above form, with $u^{(1)} = 1$, while all $V_s^{(1)}$ are zero.

The possibility or the impossibility of the k-th step in the process of determining the functions $u^{(k)}$, $v_s^{(k)}$ depends on the possibility or impossibility of a periodic solution for $u^{(k)}$

$$u^{(k)} = \int_0^{\theta} U^{(k)} d\theta.$$

Assume that all the functions $u^{(\mu)}$, $v_s^{(\mu)}$, for which μ is less than some number m have been found and are periodic functions of θ. Then the function $U^{(m)}$ may be represented in the form of a finite series of sines and cosines of whole multiples of the angle θ, and if in this series there is no constant term, the functions $u^{(m)}$, and consequently all $u_s^{(m)}$ will be periodic. In the contrary case the function $u^{(m)}$ will have the form $u^{(m)} = g\theta + v$, where g is some non-zero

constant, while v is the finite series of sines and cosines of multiples of the angle θ. The functions $v_s^{(m)}$ contain secular terms.

Let us assume the existence of this last case.

Assuming that the calculation has been carried out in such a manner that all the functions $u^{(p)}$, $u_s^{(p)}$, $v\,(p < m)$ are real for real θ, we transform equations (30) by means of the substitution

$$r = z + u^{(2)}z^2 + \ldots + u^{(m-1)}z^{m-1} + vz^m,$$
$$x_s = u_s^{(1)}z + u_s^{(2)}z^2 + \ldots + u_s^{(m-1)}z^{m-1} + z_s,$$

where $z, z_1, \ldots, z_n$ are new variables in place of the previous $r, x_1, \ldots, x_n$.

The transformed equations will have the form

$$\left.\begin{aligned} \frac{dz}{d\theta} &= zZ, \\ \frac{dz_s}{d\theta} &= q_{s1}z_1 + \ldots + q_{sn}z_n + Z_s, \end{aligned}\right\} \tag{32}$$

where

$$zZ = \frac{rR - U^{(2)}z^2 - \ldots - U^{(m-1)}z^{m-1} - (U^{(m)} - g)z^m}{1 + 2u^{(2)}z + \ldots + (m-1)u^{(m-1)}z^{m-2} + mvz^{m-1}},$$
$$Z_s = Q_s - U_s^{(1)}z - \ldots - U_s^{(m-1)}z^{m-1} +$$
$$+ (a_s \cos\theta + b_s \sin\theta)vz^m -$$
$$- [u_s^{(1)} + 2u_s^{(2)}z + \ldots + (m-1)u_s^{(m-1)}z^{m-2}]zZ.$$

The functions Z, Z_s are holomorphic functions of the variables $z, z_1, \ldots, z_n$ with coefficients depending on the real values of θ in the form of finite series of sines and cosines of whole multiples of θ. These functions vanish when all z_s and z become equal to zero; then Z_s do not contain first-order terms in their expansions.

If by $Z^{(0)}$, $Z_s^{(0)}$ are denoted the functions Z, Z_s, when in these we set

$$z_1 = 0, \ldots, z_n = 0,$$

the expansion $Z^{(0)}$ in integral powers of z will begin with the $(m-1)$-st degree with constant coefficient g and the

expansions of the functions $Z_s^{(0)}$ will contain z in powers not lower than the m-th.

Thus, equations (32) have all the properties which we found in the study of stability in the case of a single zero root.

Criteria of Stability and Instability

56[37L]. Let

$$zZ = gz^m + P^{(1)} z + \ldots + P^{(m-1)} z^{m-1} + R,$$
$$Z_s = P_s^{(1)} z + \ldots + P_s^{(m-1)} z^{m-1} + R_s,$$

where $P^{(j)}$, $P_s^{(j)}$ are linear forms of the variables $z_1, \ldots, z_n$ with periodic coefficients in θ; R is a holomorphic function of the variables z, z_s, whose expansion has similar coefficients, where $R^{(0)}$ contains z in not lower than the $(m+1)$-st power and the terms which are linear with respect to $z_1, \ldots, z_n$ contain z in powers not lower than the m-th; R_s are holomorphic functions of z, z_s, in the expansions of which terms linear with regard to $z_1, \ldots, z_n$ may contain z only in degrees higher than $(m-1)$-st; for $z_1 = \ldots = z_n = 0$ the functions R_s contain z in degrees not lower than the m-th.

To solve the problem of stability we consider the function

$$V = z + W + W^{(1)} z + \ldots + W^{(m-1)} z^{m-1},$$

where $W^{(j)}$ are linear forms and W is a quadratic form of the variables $z_1, \ldots, z_n$. The coefficients will be assumed constant in W and certain periodic functions of θ in $W^{(j)}$.

Let us arrange the coefficients in the forms $W^{(j)}$, W so that the total derivative of the function V with regard to θ is sign-definite for $z \geqslant 0$

$$\frac{dV}{d\theta} = g(z^m + z_1^2 + \ldots + z_n^2) + S,$$

where

$$S = vz^m + \sum_{s,r} v_{sr} z_s z_r,$$

v, v_{sr} being some holomorphic functions of $z, z_1, \ldots, z_n$, which vanish when all variables z, z_s become zero.

From equations (32) we have:

$$\frac{dV}{d\theta} = zZ + (W^{(1)} + 2W^{(2)}z + \ldots + (m-1)\,W^{(m-1)}z^{m-2})\,zZ + \\ + \sum_s \frac{\partial W}{\partial z_s}(q_{s1}z_1 + \ldots + q_{sn}z_n + Z_s) + \\ + \sum_s \sum_{r=1}^{m-1} z^r \frac{\partial W^{(r)}}{\partial z_s}(q_{s1}z_1 + \ldots + q_{sn}z_n + Z_s) + \sum_{r=1}^{m-1} \frac{\partial W^{(r)}}{\partial \theta} z^r.$$

If we substitute here for the functions Z, Z_s, we obtain:

$$\frac{dV}{d\theta} = [1 + W^{(1)} + 2W^{(2)}z + \ldots + (m-1)\,W^{(m-1)}z^{m-2}] \times \\ \times (gz^m + P^{(1)}z + \ldots + P^{(m-1)}z^{m-1} + R) + \\ + \sum_s \frac{\partial W}{\partial z_s}(q_{s1}z_1 + \ldots + q_{sn}z_n + Z_s) + \\ + \sum_s \sum_{r=1}^{m-1} z^r \frac{\partial W^{(r)}}{\partial z_s}(q_{s1}z_1 + \ldots + q_{sn}z_n + \\ + P_s^{(1)}z + \ldots + P_s^{(m-1)}z^{m-1} + R_s) + \sum_{r=1}^{m-1} z^r \frac{\partial W^{(r)}}{\partial \theta}.$$

Comparing this expression with the required form of the total derivative, we find the functions W, $W^{(r)}$ are given by equations

$$\sum_s \frac{\partial W}{\partial z_s}(q_{s1}z_1 + \ldots + q_{sn}z_n) = g(z_1^2 + \ldots + z_n^2),$$

$$\frac{\partial W^{(r)}}{\partial \theta} + \sum_s \frac{\partial W^{(r)}}{\partial z_s}(q_{s1}z_1 + \ldots + q_{sn}z_n) + P^{(r)} + \\ + \sum_s \left(P_s^{(1)} \frac{\partial W^{(r-1)}}{\partial z_s} + \ldots + P_s^{(r-1)} \frac{\partial W^{(1)}}{\partial z_s}\right) = 0 \\ (r = 1, \ldots, m-1).$$

The functions W are defined uniquely since all roots of the equation

$$\|q_{ij} - \delta_{ij}\varkappa\| = 0$$

have negative real parts (section 34). The functions

$$W^{(1)}, \ldots, W^{(m-1)}$$

are defined recursively in the order of increasing index r. Assuming that all preceding functions $W^{(r)}$ up to some $W^{(r)}$

have been determined, we have for determining $W^{(r)}$ the equation

$$\frac{\partial W^{(r)}}{\partial \theta} + \sum_s \frac{\partial W^{(r)}}{\partial z_s}(q_{s1}z_1 + \ldots + q_{sn}z_n) = A_1 z_1 + \ldots + A_n z_n,$$

in which the A are known finite series of sines and cosines in whole multiples of θ. To determine the coefficients a in the form

$$W^{(r)} = a_1 z_1 + \ldots + a_n z_n$$

we obtain the following system of equations:

$$\frac{da_s}{d\theta} + q_{1s}a_1 + \ldots + q_{ns}a_n = A_s \qquad (s = 1, \ldots, n).$$

Since the characteristic equation of this system does not have purely imaginary roots, there exists only one such solution in which all a are finite series of sines and cosines of whole multiples of θ.

For this definition of W, $W^{(r)}$ the magnitude of the derivative $\frac{dV}{d\theta}$, under the condition $z \geqslant 0$, is a sign-definite function with regard to $z, z_1, \ldots, z_n$ for sufficiently small numerical values of $z, z_1, \ldots, z_n$ and has the sign of the constant g.

If g is negative, then for $z > 0$ the function V will be positive-definite with respect to $z, z_1, \ldots, z_n$, since in accordance with section 35 the function W will be positive-definite with regard to the variables $z_1, \ldots, z_n$.

On the other hand, if $g > 0$, then W will be negative-definite with respect to the variables $z_1, \ldots, z_n$, the function V may be made positive by the choice $z > 0$, $z_1 = 0, \ldots, z_n = 0$, i.e. it will have the same sign as g; the function V has an infinitely small upper bound.

From this, by reason of Lyapunov's theorems on stability and instability, we must conclude that in the case when g is negative the unperturbed motion is stable with respect to $z, z_1, \ldots, z_n$ and thus with respect to $r, x_1, \ldots, x_n$ also and every sufficiently close perturbed motion tends to it asymptotically, while in the case of positive g the unperturbed motion is unstable.

57[36L]. For actual calculation of the constant g it is

not always necessary to use this method. For this purpose it is preferable to consider directly equation (29).

Let us introduce the new independent variable τ in place of t:

$$t - t_0 = \frac{\tau}{\lambda}(1 + h_2 c^2 + h_3 c^3 + \ldots);$$

the coefficients h are so far undefined; c is a certain constant.

We choose the values of the constants h so that the transformed equations (29)

$$\frac{dx}{d\tau} = \left(-y + \frac{X}{\lambda}\right)(1 + h_2 c^2 + \ldots),$$

$$\frac{dy}{d\tau} = \left(x + \frac{Y}{\lambda}\right)(1 + h_2 c^2 + \ldots),$$

$$\frac{dx_s}{d\tau} = \left(q_{s1}x_1 + \ldots + q_{sn}x_n + a_s x + b_s y + \frac{X_s}{\lambda}\right) \times \times (1 + h_2 c^2 + \ldots),$$

(where q_{sr}, a_s, b_s have the previous values of section 54) satisfy the series

$$x = x^{(1)}c + x^{(2)}c^2 + \ldots,$$
$$y = y^{(1)}c + y^{(2)}c^2 + \ldots,$$
$$x_s = x_s^{(1)}c + x_s^{(2)}c^2 + \ldots,$$

in which all functions $x^{(\mu)}$, $y^{(\mu)}$, $x_s^{(\mu)}$, are periodic functions of τ with common period 2π and do not depend on the constant c.

For the successive determination of the functions $x^{(\mu)}$, $y^{(\mu)}$, $x_s^{(\mu)}$ we obtain the equations

$$\left.\begin{aligned} \frac{dx^{(r)}}{d\tau} &= -y^{(r)} - h_{r-1}y^{(1)} + X^{(r)}, \\ \frac{dy^{(r)}}{d\tau} &= x^{(r)} + h_{r-1}x^{(1)} + Y^{(r)}, \\ \frac{dx_s^{(r)}}{d\tau} &= q_{s1}x_1^{(r)} + \ldots + q_{sn}x_n^{(r)} + X_s^{(r)} \\ &(s = 1, \ldots, n;\ r = 1, 2, \ldots), \end{aligned}\right\} \tag{33}$$

where $X^{(r)}$, $Y^{(r)}$ are certain known whole rational functions

of all $x^{(\mu)}$, $y^{(\mu)}$, $x_s^{(\mu)}$, having the index μ less than r and will depend on all constants h_j for which $j < r-1$. The functions $X_s^{(r)}$ will further depend on $x^{(r)}$, $y^{(r)}$ and h_{r-1}. Here we must put:

$$h_0 = 0, \quad h_1 = 0, \quad X^{(1)} = 0, \quad Y^{(1)} = 0.$$

If we assume that from the equations for $r = 1, \ldots, m-1$ we have calculated successively all functions $x^{(\mu)}$, $y^{(\mu)}$, $x_s^{(\mu)}$ for which $\mu < m$, and all constants h_j for which $j < m-1$, then to determine the functions $x^{(m)}$, $y^{(m)}$, $x_s^{(m)}$ and the constant h_{m-1} we will have from equations (33) a system corresponding to $r = m$. From this it is evident that if it be possible to find periodic $x^{(m)}$, $y^{(m)}$, then the periodic functions $x_s^{(m)}$ may always be found since all roots of the equation

$$\| q_{ij} - \delta_{ij} x \| = 0$$

have negative real parts and, consequently, the equations ($s = 1, \ldots, n$; $r = m$), under the given assumptions, will not correspond to resonance.

It is always possible to satisfy the equations obtained for determining $x^{(1)}$, $y^{(1)}$ by the assumption

$$x^{(1)} = \cos \tau, \quad y^{(1)} = \sin \tau.$$

Further calculations of $x^{(\mu)}$, $y^{(\mu)}$ may be carried out so that they become zero for $\tau = 0$. Examining this assumption, we note that $X^{(m)}$, $Y^{(m)}$ are then obtained in the form of a finite series of sines and cosines of whole multiples of τ:

$$X^{(m)} = A_1 \cos \tau + A_2 \sin \tau + \ldots,$$
$$Y^{(m)} = B_1 \cos \tau + B_2 \sin \tau + \ldots,$$

where A_1, A_2, B_1, B_2, ... are certain known constants.

Assuming

$$x^{(m)} = a_1 \cos \tau + a_2 \sin \tau + \ldots,$$
$$y^{(m)} = b_1 \cos \tau + b_2 \sin \tau + \ldots,$$

from equations (33), corresponding to $r = m$, we obtain the relationships

$$a_2 + b_1 = A_1, \quad -a_1 + b_2 + h_{m-1} = A_2,$$
$$-a_2 - b_1 = B_2, \quad -a_1 + b_2 - h_{m-1} = B_1.$$

to determine the constants a_1, a_2, b_1, b_3, h_{m-1}. These relationships will be simultaneously valid only if

$$A_1 + B_2 = 0.$$

When this last condition is satisfied, the functions $x^{(m)}$, $y^{(m)}$ may be determined, since in finding the coefficients a, b in the series $x^{(m)}$, $y^{(m)}$ we find a singularity (in the form of the possible appearance of secular terms) only for terms containing the sine or cosine of the first multiple of τ. For the complete determination of all coefficients of the expansion $x^{(m)}$, $y^{(m)}$ it will be necessary to use the condition $x^{(m)}(0) = 0$, $y^{(m)}(0) = 0$.

If for some index m the condition $A_1 + B_2 = 0$ is not satisfied, the required solution is not possible. It is possible to show that in this case the number m and the constant

$$g = \frac{A_1 + B_2}{2}$$

would be the same as those considered in the preceding paragraph.

We shall not consider other possible methods of determining the sign of the constant g.

58. Example. Let the differential equations of perturbed motion lead to a single equation of the following form:

$$\frac{d^2x}{dt^2} + x = F\left(x, \frac{dx}{dt}\right),$$

where F is a holomorphic function of these arguments, not containing in its expansion terms of lower than second order in the quantities x and $x' = \frac{dx}{dt}$, with regard to which we want to determine the stability of the unperturbed motion $(x = 0, x' = 0)$.

In the function F we separate terms with odd and even powers of the velocity x'

$$F = ax' f(x, x'^2) + g(x, x'^2),$$

where f, g are holomorphic functions of their arguments; f does not contain terms of lower than first order, and g does not contain terms lower than second order in x. We

shall assume that the functions f and g do not depend on the constant a and the first term $f_k(x, x'^2)$ in the expansion $f = f_k + f_{k+1} + \dots$ is positive. This problem is of interest; certain practical problems lead to it, in particular the question of stability of the equilibrium position $(x = 0)$ of a mechanical system under the action of a generalized force defined by the equation.

Putting

$$x = r \sin \theta, \quad x' = r \cos \theta,$$

we find from our equation

$$\frac{dr}{d\theta} = R_2 r^2 + R_3 r^3 + \dots,$$

where all the R denote functions of the same θ. Here all the functions

$$R_2, R_3, \dots, R_k, R_{k+1} - a \cos^2 \theta f_k(\sin \theta, \cos^2 \theta)$$

will be independent of the constant a. Therefore in finding a solution of the last equation in the form of a series

$$r = c + u_2 c^2 + u_3 c^3 + \dots,$$

expanded in ascending powers of the arbitrary constant c, all functions

$$u_2, u_3, \dots, u_k, u_{k+1} - a \int_0^\theta \cos^2 \theta f_k(\sin \theta, \cos^2 \theta)\, d\theta$$

are obtained independent of a.

But if a is zero, the proposed equation will admit a holomorphic integral not depending on t, in which the set of terms of lowest order will be $x^2 + x'^2$. Actually, from the equation obtained after elimination of t,

$$\frac{d(x^2 + x'^2)}{dx} = 2g(x, x'^2),$$

in which the right-hand side

$$2g(x, x'^2) = \varphi(x, x^2 + x'^2)$$

is a holomorphic function of the quantities x and $x'^2 + x^2$. From a well known theorem we find:

$$x^2 + x'^2 = c + \psi(x, c),$$

where ψ will be a holomorphic function of x and c, which vanishes for $x=0$; c is the value of x'^2 when x is zero. The last equation indicates directly the existence of the integral of the above character.

A sufficiently small absolute magnitude of this sign-definite integral will correspond to some closed orbits in the plane x, x' or certain periodic solutions of the above equation, for $a=0$.

It is also possible to verify the existence of these periodic solutions directly, since the equations for x and x' are invariant in the substitution of t by $-t$ and x' by $-x'$.

Therefore the functions under consideration, independent of a,

$$u_2,\ u_3,\ \ldots,\ u_k,\ u_{k+1} - a\int_0^{\theta} \cos^2\theta f_k(\sin\theta,\ \cos^2\theta)\,d\theta$$

will all be periodic and, consequently, if a is not zero, the constant g is found from the formula

$$g=\frac{a}{2\pi}\int_0^{2\pi} \cos^2\theta f_k(\sin\theta,\ \cos^2\theta)\,d\theta.$$

If f_k is positive (k even), then for $a>0$ the unperturbed motion is unstable, while for $a<0$ it is stable.

It may be remarked that the stability of the unperturbed motion for $a=0$ is obvious, since the above sign-definite integral satisfies the conditions of Lyapunov's stability theorem.

Example. Consider the system of differential equations

$$\frac{dx}{dt}=-y-\frac{a}{2}(x^2+y^2)x,$$
$$\frac{dy}{dt}=x-\frac{a}{2}(x^2+y^2)y,$$
$$\frac{dx_1}{dt}=-x_1+(x^2+y^2).$$

We shall carry out the investigation according to section 57. Putting

$$t-t_0=(1+h_2c^2+h_3c^3+\ldots)\tau,$$

we shall look for a periodic solution in the form of a series

in ascending powers of the arbitrary constant c

$$x = x^{(1)}c + x^{(2)}c^2 + \dots,$$
$$y = y^{(1)}c + y^{(2)}c^2 + \dots,$$
$$x_1 = x_1^{(1)}c + x_1^{(2)}c^2 + \dots$$

By the recursive process described in section 57, we find

$$x^{(1)} = \cos\tau, \quad y^{(1)} = \sin\tau, \quad x_1^{(1)} = 0,$$
$$x^{(2)} = 0, \quad y^{(2)} = 0, \quad x_1^{(2)} = 1,$$

while for $x^{(3)}$, $y^{(3)}$ we obtain the following equations:

$$\frac{dx^{(3)}}{d\tau} = -y^{(3)} - h_2 \sin\tau - \frac{a}{2}\cos\tau,$$
$$\frac{dy^{(3)}}{d\tau} = x^{(3)} + h_2 \cos\tau - \frac{a}{2}\sin\tau,$$

from which, apart from the impossibility of the required periodic solution for non-zero a, we obtain for the constant g the equality

$$g = \frac{-a}{2}.$$

Consequently, the unperturbed motion will be stable for positive a and unstable for negative a.

One remark of Lyapunov's should be noted. The question of stability with respect to the variables x, y is settled directly by Lyapunov's theorems No. 8, 13 and consideration of the function $V = x^2 + y^2$. However it would be erroneous to assume that the problem of stability is then solved, since it is always possible to find individual series $x_s = f_s(x, y)$, expanded in positive integral powers of x, y and not containing constant terms, such that the equations (29) for x_s are formally satisfied. In this case the equation

$$\left[-y - \frac{a}{2}x(x^2+y^2)\right]\frac{\partial x_1}{\partial x} + \left[x - \frac{a}{2}y(x^2+y^2)\right]\frac{\partial x_1}{\partial y} = -x_1 + x^3 + y^3$$

formally defines the corresponding series

$$(x^2+y^2) + a(x^2+y^2)^2 + 1\cdot 2\cdot a^2(x^2+y^2)^3 + 1\cdot 2\cdot 3\cdot a^3(x^2+y^2)^4 + \dots,$$

which is divergent for all non-zero x and y.

59[38L]. In the present case of a pair of purely imaginary roots, as we have seen, solution of the question of stability is connected with the question of the possibility of some periodic solution of system (30). But, unfortunately, all the methods which may be proposed for solving this last question lead to a solution only for the case where a negative answer is to be obtained for it. If this periodic solution exists, the unperturbed motion is stable.

Let us assume that somehow we have succeeded in establishing the existence of the required periodic solution and thus the existence and convergence of the series (31).

Then the transformation

$$r = z + u^{(2)}z^{(2)} + u^{(3)}z^{(3)} + \dots,$$
$$x_s = z_s + u_s^{(1)}z + u_s^{(2)}z^2 + \dots$$

leads to equations of the form (32), in which the functions $Z^{(\,)}$, $Z_s^{(\,)}$ vanish. We come to a case similar to that considered in section 51.

In this case system (32) has a complete integral relationship with a single arbitrary constant c:

$$z = c + \sum_m \sum_k P_m^{(k)} c^k \qquad (m,\ k = 1,\ 2,\ \dots), \tag{34}$$

where $P_m^{(k)}$ are homogeneous forms of the m-th degree in the variables $z_1, \dots, z_n$ while the coefficients of the form $P_m^{(k)}$ are finite series of sines and cosines of whole multiples of θ.

In fact, in the equations defining z,

$$\sum_s (q_{s1}z_1 + \dots + q_{sn}z_n)\frac{\partial z}{\partial z_s} + \frac{\partial z}{\partial \theta} = zZ - \sum_s Z_s \frac{\partial z}{\partial z_s}$$

there will not be terms in the right-hand side that are independent of z_s, since $Z^{(0)}$, $Z_s^{(0)}$ are zero. Let the result of substituting z in the right-hand side be represented by the form

$$-\sum_m \sum_k Q_m^{(k)} c^k;$$

$Q_m^{(k)}$ denotes the form of the m-th power of the variables z_s,

which depends on those $P_j^{(i)}$ for which $i+j<m+k$.

To determine the forms $P_m^{(k)}$ we have the equations

$$\sum_s (q_{s1}z_1+\ldots+q_{sn}z_n)\frac{\partial P_m^{(k)}}{\partial z_s}+\frac{\partial P_m^{(k)}}{\partial \theta}=-Q_m^{(k)}.$$

Assuming that all forms $P_j^{(i)}$ have been found for which $i+j<m+k$ and that they have the above described periodic coefficients with regard to θ, we note that $Q_m^{(k)}$ is a form in the same coefficients. To determine the coefficients of the form $P_m^{(k)}$ we obtain an inhomogeneous system of ordinary linear differential equations for which, from the discussion in section 34, the characteristic equation will have as roots the expressions of the form $(m_1\varkappa_1+\ldots+m_n\varkappa_n)$, where m_s are certain non-negative whole numbers, whose sum is m, while $\varkappa_1,\ldots,\varkappa_n$ are roots of the equation

$$\|q_{ij}-\delta_{ij}\varkappa\|=0;$$

certain terms will be periodic with regard to θ. Such a system (since all roots $\varkappa_s$ have negative real parts) always has one (and only one) periodic solution.

All the coefficients $P_m^{(k)}$ and the series for z are determined recursively. We shall prove the convergence of the latter series for sufficiently small absolute values of $c, z_1, \ldots, z_n$.

The system of differential equations for the coefficients of form $P_{m_s}^{(k_s)}$ are brought to canonical form (section 30). The coefficients of the form $P_{m_s}^{(k_s)}$ will be connected with the new canonical quantities A by certain formulæ of linear transformation, generally with complex coefficients of bounded modulus. In the process of recursive determination of the coefficients A we shall consider the equation

$$\frac{dA}{d\theta}+(m_1\varkappa_1+\ldots+m_n\varkappa_n)A=-B,$$

where B, according to the formulae of the above linear transformation, depends on the coefficients of the form $Q_m^{(k)}$ and further, possibly on the preceding A, in the form of an additional term. The function B will be a periodic function of θ with bounded modulus.

From this we find

$$A = e^{-(m_1 x_1 + \ldots + m_n x_n)\theta} \int_0^{\infty} e^{(m_1 z_1 + \ldots + m_n x_n)\theta} B\, d\theta.$$

From this expression for A it follows that we obtain somewhat greater upper bounds for the moduli of the coefficients of the forms $P_m^{(k)}$, equally valid for all real values of θ. Consequently z will be a holomorphic function of the quantities $z_1, \ldots, z_n, c$ for sufficiently small values of these quantities and for all real values of θ.

Let us return to our problem. We shall replace the first of equations (32) by a total integral relationship (34). The remaining equations of (32), after substituting for z from (34), take the form

$$\frac{dz_s}{d\theta} = (q_{s1} + c_{s1}) z_1 + \ldots + (q_{sn} + c_{sn}) z_n + Z_s',$$

where c_{sr} are holomorphic functions of the constant c, which vanish when c becomes zero, with coefficients which are real periodic functions of θ; Z_s' are holomorphic functions of the quantity $z_1, \ldots, z_n, c$, the expansions of which begin with terms of not less than second order with respect to the variable z_s and have periodic real coefficients in θ. All these functions are holomorphic for all real values of θ.

As in section 51 we consider the negative-definite quadratic form W, defined by the equation

$$\sum_s (q_{s1} z_1 + \ldots + q_{sn} z_n) \frac{\partial W}{\partial z_s} = z_1^2 + \ldots + z_n^2.$$

The total derivative with respect to θ of this form W taken in accordance with the last equations will be:

$$\frac{dW}{d\theta} = z_1^2 + \ldots + z_n^2 + \\ + \sum_s (c_{s1} z_1 + \ldots + c_{sn} z_n) \frac{\partial W}{\partial z_s} + \sum_s Z_s' \frac{\partial W}{\partial z_s}.$$

Since c_{sr} are holomorphic functions of c for all real values of θ, which vanish for $c = 0$, it is possible to determine a certain positive constant h such that, for c numerically not exceeding h, the quadratic form

$$z_1^2 + \ldots + z_n^2 + \sum_s (c_{s1} z_1 + \ldots + c_{sn} z_n) \frac{\partial W}{\partial z_s}$$

will be positive-definite with respect to z_s for all real values of θ. Then for sufficiently small absolute magnitudes of z_s say

$$\sum z_s^2 \leqslant R$$

the derivative $\frac{dW}{d\theta}$ will be positive-definite. And this, from Lyapunov's theorem (section 11), proves that the unperturbed motion will be stable with respect to the variables $z_1, \ldots, z_n$, independently of the value of c, if their absolute values are less than h; thus a sufficiently close perturbed motion will tend asymptotically to the motion

$$z = c, \; z_1 = 0, \; \ldots, \; z_n = 0.$$

The positive-definite function $V = -W$ has constant coefficients independent of c. Let l be the exact upper bound of the function V on the sphere $z_1^2 + \ldots + z_n^2 = A$. The function V, which does not depend explicity on θ, has an infinitely small upper bound and therefore for l a number ε can be found such that for

$$\sum_s z_s^2 < \varepsilon$$

the inequality

$$V < l.$$

will be valid.

Thus, the definite number ε is independent of c. From this, we see that, for all values of $|c|$, less than h, and for each positive number $A \leqslant R$ the above-defined number ε (which is independent of c) has the property that for initial perturbations subject to the condition

$$\sum_s z_{s0}^2 < \varepsilon,$$

for arbitrary real θ, the inequality

$$\sum_s z_s^2 < A.$$

will be satisfied.

This proves the stability of the unperturbed motion with respect to the quantities $z_1, \ldots, z_n, c$, which is equivalent to

its stability with respect to $z, z_1, \ldots, z_n$ and thus to the initial variables $x, y, x_1, \ldots, x_n$.

60[39, 41L]. Example. Let us consider the differential equations of perturbed motion

$$\frac{dx}{dt} = -\lambda y + X, \quad \frac{dy}{dt} = \lambda x + Y,$$

where X, Y are holomorphic functions of x, y, their expansions beginning with terms of at least second order, and satisfying the equation

$$\frac{\partial X}{\partial y} + \frac{\partial Y}{\partial x} = 0.$$

In accordance with the last relationship, a certain holomorphic function $\psi(x, y)$ exists, its expansion beginning with terms of third order, such that

$$X = -\frac{\partial \psi}{\partial y}, \quad Y = \frac{\partial \psi}{dx}.$$

The assumed equations are canonical and will have the following integral:

$$x^2 + y^2 + \frac{2}{\lambda}\psi(x, y) = c,$$

where c is a certain constant, necessarily positive for sufficiently small absolute magnitudes of x, y.

Substituting $x = r \cos \theta, y = r \sin \theta$ and solving the integral obtained with respect to r, we obtain an expression for r in the form of a periodic function of θ. Consequently, the unperturbed motion will be stable (section 59);

This may also be found from Lyapunov's general theorem on stability, if for the function V be taken the left-hand side of the above positive-definite integral.

Example. The equations

$$\frac{dx}{dt} + y = \alpha yz, \quad \frac{dy}{dt} - x = \beta xz, \quad \frac{dz}{dt} + kz = \gamma xy,$$

are given, in which k denotes a positive constant and α, β, γ any convenient real constants.

We shall look for a periodic solution using the method described in section 57. Noting that the assumed equations

are invariant with substitution of x by $-x$ and y by $-y$, we put

$$x = c\cos\tau + x^{(3)}c^3 + x^{(5)}c^5 + \dots,$$
$$y = c\sin\tau + y^{(3)}c^3 + y^{(5)}c^5 + \dots,$$
$$z = z^{(2)}c^2 + z^{(4)}c^4 + \dots$$

To determine the function $z^{(2)}$ we have the equation

$$\frac{dz^{(2)}}{d\tau} + kz^{(2)} = \frac{\gamma}{2}\sin 2\tau,$$

which has a periodic solution

$$z^{(2)} = \frac{\gamma}{2(k^2+4)}(k\sin 2\tau - 2\cos 2\tau).$$

To determine $x^{(3)}$, $y^{(3)}$ we have the equations

$$\frac{dx^{(3)}}{d\tau} + y^{(3)} = -h_2\sin\tau + \alpha z^{(2)}\sin\tau,$$
$$\frac{dy^{(3)}}{d\tau} - x^{(3)} = h_2\cos\tau + \beta z^{(2)}\cos\tau.$$

Substituting in the right-hand sides of these equations the expression found for $z^{(2)}$, we have, in the expansion of the expressions $X^{(3)} = \alpha z^{(2)}\sin\tau$ and $Y^{(3)} = \beta z^{(2)}\cos\tau$ in terms of sines and cosines of multiple angles, the following terms which are of interest:

$$X^{(3)} = \frac{\alpha\gamma k}{4(k^2+4)}\cos\tau + \dots,\quad Y^{(3)} = \frac{\beta\gamma k}{4(k^2+4)}\sin\tau + \dots,$$

from which

$$g = \frac{(\alpha+\beta)\gamma k}{8(k^2+4)}.$$

Therefore (section 56), if $(\alpha+\beta)\gamma < 0$, the unperturbed motion is stable and if $(\alpha+\beta)\gamma > 0$, the unperturbed motion is unstable. If $(\alpha+\beta)\gamma = 0$ the proposed system of equations has the integral

$$x^2 + y^2 = \text{const},$$

and consequently, if we substitute $x = r\cos\theta$, $y = r\sin\theta$, r will be a periodic function of θ ($r = c$). In section 55 we established that if $u^{(\nu)}$ is periodic (in the present case all $u^{(\nu)} = 0$ for $\nu = 2, 3\dots$), then the required periodic solution (31) will always exist formally (it may be proved that the series (31) will be convergent). We should conclude from

section (39) that the unperturbed motion will be stable for $(\alpha + \beta)\gamma = 0$.

Note. The critical case when the equations of the first approximations have constant coefficients while the characteristic equation has a double zero root with a multiple elementary divisor has been considered by Lyapunov[2] for $n = 2$. Other critical cases were examined by Malkin[3] and Kamenkov[4]. These cases, which are of specialised interest, will not be considered here.

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CHAPTER 9

TRANSIENT MOTION

The Characteristic Numbers of Functions

The methods of investigating the stability of transient motion have been developed less fully than the methods proposed for steady-state motion. The existence theorems available, while not effectively solving the question, give (in a certain sense) a clear idea what may or what should occur, and this, in individual problems, is sufficient for their complete solution.

61[6L]. We shall consider functions x of a real variable t, defined for all t greater than or equal to t_0. Here we shall consider only such functions for which upper bounds exist for the moduli as t varies from t_0 to some arbitrarily given number $T > t_0$.

A function is termed bounded if its modulus for $t > t_0$ remains always less than some constant. A function the modulus of which may be made greater than any given arbitrarily great positive quantity is termed unbounded. A bounded function which tends to zero with unlimited increase in t is termed "vanishing".

From these definitions it follows directly that:

if x is a bounded function of t, then $xe^{-\lambda t}$ for all positive λ is a vanishing function;

if x is not a vanishing function of t, then $xe^{\lambda t}$ for all positive λ is an unbounded function.

Lemma. If

$$z = xe^{\lambda t}$$

represents a vanishing function for $\lambda = \lambda_1$ and is unbounded for $\lambda = \lambda'$, where λ_1 and λ' are certain real constants, it

is always possible to find a real number λ_0 such that, for $\lambda = \lambda_0 + \varepsilon$, the function z will be unbounded for all positive constant ε and will be a vanishing function for all negative constant ε.

Proof. Between the numbers λ_1 and λ' it is always possible to insert two infinite series of numbers:

$$\text{non-decreasing: } \lambda_1, \lambda_2, \lambda_3, \ldots$$
$$\text{and non-increasing: } \lambda', \lambda'', \lambda''',$$

such that each number of the first series is less than each number of the second, and that for each n the function

$$xe^{\lambda_n t}$$

be a vanishing function, while the function

$$xe^{\lambda^{(n)} t}$$

for each n be unbounded and, finally, that the difference

$$\lambda^{(n)} - \lambda_n$$

for sufficiently large n may be made arbitrarily small. This may be attained by successive insertions of pairs of numbers $(\lambda_s, \lambda^{(s)})$.

These two series define a section λ_0, not less than any of the numbers of the first series and not greater than any of the numbers of the second. The number λ_0 is to be sought. Lyapunov has proposed to call this number the characteristic number of the function x.

If $xe^{\lambda t}$ is a vanishing or an unbounded function for every real value of λ, we agree to assign to the characteristic number in the first case the value $+\infty$, in the second $-\infty$.

Examples

For each non-zero constant the characteristic number is zero.

For each bounded function the characteristic number is zero.

For a polynomial of finite degree the characteristic number is zero.

For the function $e^{\pm t \sin t}$ the characteristic number is equal to - 1.

From the definition of the characteristic number it follows that if by suitable choice of t, greater than an arbitrarily given number, the quantity

$$|\lambda - f(t)|$$

may be made arbitrarily small and if for each arbitrarily small positive constant it is possible to find a number T such that

$$\lambda - f(t) < \varepsilon$$

for all $t > T$, then λ is the characteristic number of the function

$$e^{-tf(t)}.$$

62[6L]. When the characteristic numbers of given functions are finite, it is possible to prove the following propositions:

The characteristic number of the sum of two functions is equal to the smaller of the characteristic numbers of these functions, when these numbers are distinct, and not less than either of the characteristic numbers when they are equal.

In essence, let λ_1 and λ_2 be the characteristic numbers of functions x_1 and x_2 and let $\lambda_1 \leqslant \lambda_2$. Then the function

$$(x_1 + x_2)\, e^{(\lambda_1 + \varepsilon)t} = x_1 e^{(\lambda_1 + \varepsilon)t} + x_2 e^{(\lambda_2 + \eta)t},$$

where $\eta = \lambda_1 - \lambda_2 + \varepsilon$, will be a vanishing function for all negative constant ε. Therefore the characteristic number of the sum $x_1 + x_2$ is in each case not less than λ_1. If the characteristic numbers λ_1 and λ_2 are not equal, then making ε positive and satisfying the inequality

$$0 < \varepsilon < \lambda_2 - \lambda_1,$$

we find that for this η will be negative and, consequently, the first term in the right-hand side of the last equality will be unbounded and the second vanishing; in this case the characteristic number of the sum $x_1 + x_2$ will be equal to λ_1.

The characteristic number of the product of two functions is not less than the sum of their characteristic numbers.

In essence, again let λ_1 and λ_2 be the characteristic numbers of the functions x_1 and x_2. The function

$$x_1 x_2 e^{(\lambda_1+\lambda_2+\varepsilon)t} = x_1 e^{\left(\lambda_1+\frac{\varepsilon}{2}\right)t} x_2 e^{\left(\lambda_2+\frac{\varepsilon}{2}\right)t}$$

is vanishing for all negative ε. This proves the proposition.

That the characteristic number of a product may be greater than the sum of characteristic numbers of the factors is proved by the example

$$x_1 = e^{t \sin t}, \quad x_2 = e^{-t \sin t}.$$

Each of these functions has a characteristic number -1, while the characteristic number of their product is equal to zero.

Corollary. The sum of characteristic numbers of the functions x and $\frac{1}{x}$ is not greater than zero.

If

$$x = e^{-(f+i\varphi)t},$$

where $i = \sqrt{-1}$, while f and φ are certain real functions of t, in order that the sum of the characteristic numbers of the functions x and $\frac{1}{x}$ be equal to zero, it is necessary and sufficient that the function f approach a certain limit with unbounded increase of t.

The sufficiency is obvious since if the function f tends to a certain limit λ with unbounded increase in t, the latter serves as the characteristic number of the function x.

The necessity follows from the fact that if λ and $-\lambda$ are characteristic numbers of the functions x and $\frac{1}{x}$, for each given positive arbitrarily small ε, the functions

$$e^{-t(\varepsilon-\lambda+f)} \text{ and } e^{-t(\varepsilon+\lambda-f)}$$

are vanishing and the latter is possible only under the condition

$$|\lambda - f| < \varepsilon$$

for all values of t greater than a certain number.

If the sum of the characteristic numbers of the functions x and $\frac{1}{x}$ is equal to zero, the characteristic number of the product of the function x with any function y is equal to the sum of characteristic numbers of these functions.

Let λ, μ, S be characteristic numbers of functions x, y, $z = xy$. Then, applying the proposition on the characteristic number of a product to the functions

$$z = xy, \quad y = z\frac{1}{x},$$

we find

$$S \geqslant \lambda + \mu, \quad \mu \geqslant S - \lambda,$$

from which

$$S = \lambda + \mu.$$

Let x be an integrable function of t. We shall consider the integral

$$u = \int_{t_0}^{t} x\,dt,$$

if the characteristic number of the function x is negative or zero and the integral

$$u = \int_{t}^{\infty} x\,dt,$$

if this number is positive.

The characteristic number of an integral is not less than the characteristic number of the integrand.

Let λ be the characteristic number of the integrand x. If $\lambda > 0$, we consider the integral

$$u = \int_{t}^{\infty} [xe^{(\lambda-\tau_1)t}]\, e^{-(\lambda-\tau_1)t}\,dt,$$

where $0 < \tau_1 < \lambda$. The function $xe^{(\lambda-\tau_1)t}$ for any positive

constant η will be vanishing and, consequently, bounded. Denoting by M the upper limit of its modulus for $t > t_0$, we obtain:

$$|u| < M\int_t^\infty e^{-(\lambda-\eta)t}\,dt = \frac{M}{\lambda-\eta}\,e^{-(\lambda-\eta)t}.$$

Consequently, the function

$$ue^{(\lambda-\varepsilon)t}$$

will be vanishing for all ε greater than η. But η may be taken arbitrarily small. Therefore the last function is vanishing for all positive ε. But this proves the proposition for positive λ.

If $\lambda \leqslant 0$, then for arbitrary positive η

$$|u| < M\int_{t_0}^t e^{-(\lambda-\eta)t}\,dt = \frac{M}{\eta-\lambda}\,e^{-(\lambda-\eta)t} + \text{const},$$

from which

$$ue^{(\lambda-\varepsilon)t}$$

is a vanishing function for each ε greater than η and consequently for each positive ε.

Characteristic Numbers of Solutions

63[7L]. We shall consider particular solutions of the linear system of differential equations

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n \qquad (s = 1, \ldots, n) \tag{35}$$

with coefficients p_{sr} that are real, bounded, continuous functions of t.

Let us consider some solution

$$x_1, \ldots, x_n$$

of the linear differential equations (35). By the characteristic number of this solution we shall understand the smallest of the characteristic numbers of the functions entering into the solution. There can always be found a

system of n linearly independent solutions.

Theorem. Each non-trivial solution of the system of differential equations (35) has a finite characteristic number.

Proof. Let us first consider the real solution for which all x_s are real functions of t.

We introduce the new variables

$$z_s = x_s e^{\lambda t},$$

where λ denotes a certain real constant. Then the given equations (35) are transformed to the following:

$$\frac{dz_s}{dt} = p_{s1}z_1 + \cdots + (p_{ss} + \lambda)z_s + \cdots + p_{sn}z_n,$$

from which we derive

$$\frac{1}{2}\frac{d}{dt}\sum_s z_s^2 = \sum_{s,r}(p_{sr} + \delta_{sr}\lambda)z_s z_r.$$

The second part of this equation represents a certain real quadratic form of the quantities z with coefficients which depend on λ and t. Because of the assumed boundedness of the functions p_{sr}, it is always possible to find such values $\lambda = \lambda'$, for which all principal diagonal minors of the discriminant

$$\left\| \frac{p_{sr} + p_{rs}}{2} + \delta_{sr}\lambda \right\|$$

will be positive for all values of t considered. For this value of λ' the quadratic form will be positive-definite.

There are also found values $\lambda = \lambda_1$, for which the principal diagonal minors will be sign-alternating, beginning with the negative $p_{11} + \lambda_1$; for such λ_1, the quadratic form on the right-hand side of the last equation will be negative-definite.

For each $\lambda = \lambda' + \frac{\varepsilon}{2}$, where ε is an arbitrary positive constant, we have the inequality

$$\frac{d}{dt}\sum_s z_s^2 > \varepsilon \sum_s z_s^2,$$

from which by integration we derive

$$\sum_{s} z_s^2 > Ce^{\varepsilon t}$$

for each value of t considered; C denotes a certain positive constant not exceeding

$$\sum z_{s0}^2 e^{-\varepsilon t_0},$$

where z_{s0} are the initial values of the variables z_s, corresponding to the initial time t_0.

For $\lambda = \lambda_1 - \frac{\varepsilon}{2}$ we have:

$$\frac{d}{dt}\sum_{s} z_s^2 < -\varepsilon \sum_{s} z_s^2,$$

from which

$$\sum_{s} z_s^2 < C'e^{-\varepsilon t}$$

for each value of t considered; C' denotes a positive constant not less than

$$e^{\varepsilon t_0}\sum z_{s0}^2.$$

Consequently, for $\lambda = \lambda' + \frac{\varepsilon}{2}$, for arbitrary positive ε, among the functions z_s there will be found at least one unbounded function while for $\lambda = \lambda_1 - \frac{\varepsilon}{2}$ all functions z_s will be vanishing. Thus, the smallest of the characteristic numbers of the functions x_s of the considered real non-trivial solution of system (35) is not less than λ_1 and not greater than λ'.

To prove the validity of the theorem in general, it is sufficient to note that each particular complex solution

$$x_s = u_s + \sqrt{-1}\, v_s \qquad (s = 1, \ldots, n)$$

consists of two particular real solutions u_s and v_s.

Note. The theorem is easily extended to the case of complex p_{sr}, if they are continuous functions of t with bounded modulus.

Corollary. If the coefficients p_{sr} of the differential equations (35) are such that the principal diagonal minors of the determinant

$$\| p_{sr} + p_{rs} \|$$

are sign alternating, where p_{11} is negative for all values of t exceeding a certain constant t_0, the characteristic numbers of the particular solutions of such a system are all positive.

64[8L]. For equations (35), let there be found systems of linearly independent solutions

$$x_{1k}, \ldots, x_{nk} \qquad (k = 1, \ldots, n).$$

By forming suitable linear combinations of these solutions, we may derive all other complete systems of independent solutions.

Each system of independent solutions has n characteristic numbers, corresponding to the n independent particular solutions entering into it. If all these characteristic numbers are distinct, the characteristic number of any other particular solution expressed linearly in terms of these independent solutions will be equal to one of the characteristic numbers of such a system of independent solutions, in accordance with the proposition on the characteristic number of the sum (section 62). Therefore the system of equations (35) cannot have more than n non-trivial solutions, the characteristic numbers of which are all distinct.

If the system of independent solutions has identical characteristic numbers, it may occur that it is possible to find a new system of independent particular solutions, the sum of characteristic numbers of which will be greater than the corresponding sum for the initial solution. Since the number of characteristic numbers is bounded (it cannot be greater than n), there exists a complete system of particular independent solutions for which the sum of characteristic numbers of all component solutions attains its maximum value; such a system of independent solution is termed normal.

The system of independent solutions for which the characteristic numbers are distinct is obviously normal.

65[8L]. Let us consider the determinant composed of functions x_{sr} of the system of independent particular solutions

$$\Delta = \|x_{sr}\|.$$

The total derivative with respect to t, from equations (35), is

$$\Delta' = \Delta \sum_{s=1}^{n} p_{ss}.$$

Integration gives Liouville's formula

$$\Delta = Ce^{\int \Sigma p_{ss}\, dt},$$

where C is some constant.

We denote by

$$\lambda_1, \ldots, \lambda_n$$

the characteristic numbers of the system of independent solutions $x_{1k}, \ldots, x_{nk}$ $(k = 1, \ldots, n)$. The propositions on the characteristic numbers of the sum and the product (section 62), applied to the formula $\Delta = \|x_{sr}\|$, lead to the conclusion that the characteristic number Δ is not less than the sum

$$\lambda_1 + \ldots + \lambda_n.$$

In other words, the sum of characteristic numbers of a system of independent solutions of equations (35) does not exceed the characteristic number of the function

$$e^{\int \Sigma p_{ss}\, dt}.$$

Corollary. If the characteristic number of the last function is negative, then among the characteristic numbers λ_s there will be at least one which is negative.

Corollary. Every system n of independent solutions for which the sum of characteristic numbers of all solutions is equal to the characteristic number of the function

$$e^{\int \Sigma p_{ss} dt},$$

is normal.

It should be remembered that not every normal system has this property.

For example, for the system of equations

$$\frac{dx_1}{dt} = (\sin \ln t + \cos \ln t)\, x_2,$$

$$\frac{dx_2}{dt} = (\sin \ln t + \cos \ln t)\, x_1$$

the function

$$e^{\int \Sigma p_{ss} dt},$$

being constant, has the characteristic number zero while the system of independent solutions

$$x_{11} = e^t \sin \ln t, \qquad x_{21} = e^t \sin \ln t,$$

$$x_{12} = e^{-t} \sin \ln t, \qquad x_{22} = -e^{-t} \sin \ln t,$$

which is obviously normal (as none of its linear combinations can increase the characteristic number common to them), has the sum of characteristic numbers (namely -2) less than zero. The characteristic numbers of the solutions are here defined similarly to the last example in section 61; the upper limits of the functions $\pm \sin \ln t$ attained for t greater than any given pre-assigned number, are equal to $+1$.

The characteristic numbers of this system of independent solutions, which define in a certain sense the increase of the functions x_{sr} for unbounded increase of t, give directly the conditions of stability or instability of the trivial solution $x_1 = \ldots = x_n = 0$ of the system (35).

If all characteristic numbers of the system of independent solutions are positive, then all functions x_{sr} will be vanishing, and the unperturbed motion $(x_1 = \ldots = x_n = 0)$ is stable.

If among the characteristic numbers of the system of independent solutions is found at least one negative, then among the functions x_{sr} is found at least one unbounded and, consequently, the unperturbed motion will be unstable.

Thus, the problem of stability or instability for linear equations with variable coefficients leads to the problem of determining at least the sign of the smallest characteristic number of any of the complete systems of its independent particular solutions. This problem has not been solved.

Theorem. If in a system of linear equations (35) the coefficients tend to bounded limits c_{sr} for unbounded

increase in t, its lowest characteristic number coincides with the lowest characteristic number of the limiting system of equations

$$\frac{dx_s}{dt} = c_{s1}x_1 + \ldots + c_{sn}x_n \qquad (s = 1, \ldots, n).$$

Proof. Let us substitute

$$z_s = x_s e^{\eta t},$$

where η denotes some constant number. The given equations (35) are transformed to the system

$$\frac{dz_s}{dt} = p_{s1}z_1 + \ldots + (p_{ss} + \eta)z_s + \ldots + p_{sn}z_n, \tag{36}$$

while the limiting system of the theorem, with this substitution, is transformed to the limiting system for (36):

$$\frac{dz_s}{dt} = c_{s1}z_1 + \ldots + (c_{ss} + \eta)z_s + \ldots + c_{sn}z_n. \tag{37}$$

Let us denote the roots of the characteristic equation of the system (37)

$$\| c_{sr} - \delta_{sr}(\varkappa - \eta) \| = 0$$

by

$$\varkappa_1, \ldots, \varkappa_n.$$

If there do not exist any non-negative integers $m_1, \ldots, m_n$, whose sum is 2, for which the expression

$$m_1\varkappa_1 + \ldots + m_n\varkappa_n,$$

vanishes, then from section 34 a quadratic form W with constant coefficients will exist satisfying the equation

$$\sum_s \frac{\partial W}{\partial z_s}[c_{s1}z_1 + \ldots + (c_{ss} + \eta_1)z_s + \ldots + c_{sn}z_n] = -z_1^2 - \ldots - z_n^2.$$

The form W will be negative-definite if the real parts of all roots $\varkappa_s$ are negative (section 35); for certain values of the variables $z_1, \ldots, z_n$ it will take on positive values if among the roots $\varkappa_1, \ldots, \varkappa_n$ there is at least one root with positive real part.

The total derivative with regard to t of this function W by reason of equations (36) will be:

$$\frac{dW}{dt} = z_1^2 + \ldots + z_n^2 + \sum_{sr} (p_{sr} - c_{sr}) z_r \frac{\partial W}{\partial z_s}.$$

Since W represents a quadratic form with constant coefficients, a non-zero positive number ε can be found such that if the inequalities

$$|p_{sr} - c_{sr}| < \varepsilon$$

are satisfied, the right-hand side of the last relationship will represent a positive-definite quadratic form of the variables $z_1, \ldots, z_n$.

From the conditions of the theorem, the coefficients p_{sr} tend, with unbounded increase of t, to the numbers c_{sr}. Therefore for the arbitrarily small positive number ε, there can be found a value t_0 such that, for values $t > t_0$, the absolute values of all quantities $p_{sr} - c_{sr}$ will be less than ε and, consequently, for all such values of t the derivative $\frac{dW}{dt}$ will represent a positive-definite function.

But, from Lyapunov's general theorems on stability and instability, this means that if, under the assumptions made about the roots $\varkappa_1, \ldots, \varkappa_n$, the largest of their real parts is negative, then the unperturbed motions both in system (37) and in system (36) are stable; if the largest real part of the roots $\varkappa_1, \ldots, \varkappa_n$ is positive, the unperturbed motions both in system (37) and in system (36) are unstable.

The assumptions made about the roots $\varkappa_1, \ldots, \varkappa_n$ will not be satisfied for only a finite number of values of η. From the form of the substitution we have applied, we must conclude that, when $x_1, \ldots, x_n$ are taken as particular solutions of both the given equations and the limiting system, the lowest characteristic numbers of the functions $z_1, \ldots, z_n$ may equal zero simultaneously for one definite value of the constant η. But this proves the theorem.

66. If the coefficients p_{sr} have the form

$$p_{sr} = c_{sr} + \varepsilon f_{sr},$$

where ε is some parameter, f_{sr} is bounded, and the constants c_{sr} do not depend on ε, then it is possible to relate equations (35) to equations with constant coefficients

c_{sr}. We form the equation

$$\|c_{sr} - \delta_{sr}\lambda\| = 0$$

with respect to the roots λ_s, about which it was previously assumed that they satisfy the condition

$$m_1\lambda_1 + \ldots + m_n\lambda_n \neq 0$$

for arbitrary non-negative integers $m_1, \ldots, m_n$, whose sum is 2. Considering the quadratic form

$$V = \frac{1}{2}\sum a_{rs}x_r x_s,$$

which satisfies the equation

$$\sum_s (c_{s1}x_1 + \ldots + c_{sn}x_n)\frac{\partial V}{\partial x_s} = x_1^2 + \ldots + x_n^2,$$

we note that for sufficiently small $|\varepsilon|$ and positive μ, less than unity, the form

$$V' - \mu(x_1^2 + \ldots + x_n^2)$$

may be made positive for arbitrary values of the variables. Then the asymptotic stability or instability of the unperturbed motion $(x_1 = 0, \ldots, x_n = 0)$ of the system of equations with constant coefficients c_{sr} correspond to that of system (35); the quantity ε for which this correspondance exists unconditionally is defined by n inequalities

$$D_r = \begin{vmatrix} h_{11} \ldots h_{1r} \\ \cdot \quad \cdot \quad \cdot \\ h_{r1} \ldots h_{rr} \end{vmatrix} > 0 \qquad (r = 1, \ldots, n),$$

where

$$h_{rs} = \delta_{rs} + \frac{\varepsilon}{2}\sum_i (a_{ri}f_{is} + a_{si}f_{ir}).$$

In addition it is possible to note that the bounded functions f_{sr} may be functions not only of t but of the variables $x_1, \ldots, x_n$, subject always to the assumed conditions:

$$\text{for } t \geqslant t_0 \quad \sum x_s^2 \leqslant A.$$

In this case equations (35) will represent the "linearization" of certain non-linear equations while the inequalities

$D_r > 0\,(r = 1, \ldots, n)$ will define the ε and A for which the above relationship exists unconditionally. In many problems it is sufficient for the constant c_{sr} to take on the values of the coefficients p_{sr} for a certain fixed moment of time and for zero values of the variables x_i.

It is possible to indicate a further procedure for solving the problem of stability for a linear equation (35) with variable coefficients $p_{sr}(t)$.

For each value of the independent variable t the equation

$$\Delta(t) = \| p_{sr} - \delta_{sr}\lambda \| = 0$$

defines n roots $\lambda_1, \ldots, \lambda_n$, varying with time t.

If non-negative integers $m_1, \ldots, m_n$, (whose sum is equal to 2) for which the expression $m_1\lambda_1 + \ldots + m_n\lambda_n$ vanishes, cannot be found for any $t \geqslant t_0$, then for such values of t there will exist the quadratic form

$$V = \sum a_{sr}x_s x_r \qquad (a_{sr} = a_{rs})$$

with bounded coefficients a_{rs}, dependent on t, satisfying the first-order partial differential equation

$$\sum \frac{\partial V}{\partial x_s}\left(p_{s1}x_1 + \ldots + p_{sn}x_n\right) = x_1^2 + \ldots + x_n^2,$$

in which t plays the role of the parameter.

The form V will be negative if the real parts of all the roots λ_s are negative; for certain values of the variables x_s it will take on positive values if among the roots $\lambda_1, \ldots, \lambda_n$ there exists at least one root with positive real part. Its total derivative with regard to time by reason of equations (35) is

$$\frac{dV}{dt} = x_1^2 + \ldots + x_n^2 + \frac{\partial V}{\partial t}.$$

The principal diagonal minors of the discriminant of this quadratic form are

$$D_r = \begin{vmatrix} a'_{11} + 1 & \ldots & a'_{1r} \\ \ldots & \ldots & \ldots \\ a'_{r1} & \ldots & a'_{rr} + 1 \end{vmatrix} \qquad (r = 1, \ldots, n),$$

where

$$a_{rs} = \frac{da_{rs}}{dt}.$$

For all $t \geqslant t_0$ let the derivatives a'_{rs} be bounded, while no D_r is less than some positive number; the derivative V', from Sylvester's well-known criterion will then be a positive-definite quadratic form of the variables $x_1, \ldots, x_n$. For these conditions if V is a negative-definite quadratic form, the unperturbed motion is stable; if V further has an infinitely small upper bound, the stability of the unperturbed motion will be asymptotic. If the form V has an infinitely small upper bound and can take on positive values, the unperturbed motion is unstable.

It is not superfluous to remark that the stability condition may in individual cases be improved by varying the function V.

Regular Systems

67. In accordance with the proposition on the characteristic number of the product (section 62), the sum of characteristic numbers of the functions

$$e^{\int \Sigma p_{ss} dt} \quad \text{and} \quad e^{-\int \Sigma p_{ss} dt}$$

is not greater than zero. Therefore if μ denotes the characteristic number of the second of these functions, the sum S of the characteristic numbers of the solutions of the normal system cannot exceed the number $-\mu$. Here the equality $S = -\mu$ is possible only under the condition that the sum of characteristic numbers of the two functions considered is equal to zero.

The system of differential equations (35) will be termed regular if the equality

$$S + \mu = 0$$

exists; otherwise it is irregular.

The equations with constant coefficients are obviously regular.

We shall now prove Lyapunov's proposition that each system of equations (35), in which all coefficients are periodic functions of t with the same real period, is regular.

Let the proposition be valid for the system (35). By λ_k we denote the characteristic number of the solution

$$x_{1k}, \ldots, x_{nk}$$

of the normal system of its independent solutions ($k = 1, \ldots, n$). By the definition of a regular system the sum $S = \lambda_1 + \ldots + \lambda_n$ is equal to the characteristic number of the function

$$e^{-\int \Sigma p_{ss} dt}$$

taken with opposite sign.

We denote by Δ the determinant, formed of the functions x_{sr} of the normal system of solutions of equations (35)

$$\Delta = \| x_{sr} \|;$$

the minor (inclusive of sign) corresponding to the element x_{ij} will be denoted by Δ_{ij}.

Let us consider the function

$$y_{sr} = \frac{\Delta_{sr}}{\Delta} \qquad (s = 1, \ldots, n).$$

By direct differentiation we verify that for each fixed r the functions $y_s = y_{sr}$ satisfy the adjoint system of linear differential equations

$$\frac{dy_s}{dt} + p_{1s}y_1 + \ldots + p_{ns}y_n = 0 \qquad (s = 1, \ldots, n).$$

The solutions y_{sr} of the adjoint and x_{sr} of the given systems satisfy the obvious relationships

$$\sum_s x_{sk} y_{sr} = \delta_{kr}.$$

Let us denote by μ_r the characteristic number of the solution $y_{1r}, \ldots, y_{nr}$ of the adjoint system of equations. From the above considerations for $k = r$ we have:

$$\mu_r + \lambda_r \leqslant 0.$$

If the given system of equations is regular, then from the formula

$$y_{sr} = \frac{\Delta_{sr}}{\Delta} = Ce^{-\int \Sigma p_{ss} dt} \Delta_{sr} \quad (s = 1, \ldots, n)$$

in accordance with the theorem on characteristic numbers

of products and sums we have

$$\text{char. number } y_{sr} \geqslant -\lambda_r \qquad (s = 1, \ldots, n).$$

This inequality is valid for any index s, and consequently, also for that value corresponding to that one of the functions $y_{1r}, \ldots, y_{nr}$, with lowest characteristic number μ_r, which by definition is taken as the characteristic number of the solution $y_{1r}, \ldots, y_{nr}$:

$$\mu_r \geqslant -\lambda_r.$$

This inequality together with the one previously established leads to the relationship

$$\mu_r + \lambda_r = 0 \qquad (r = 1, \ldots, n).$$

From this the sum of the characteristic numbers $\mu_1 + \cdots + \mu_n$ of the adjoint system is equal to the characteristic number of the function

$$e^{\int \Sigma p_{ss}\, dt}.$$

taken with reversed sign. But this proves that the system adjoint to the regular system of equations is also regular while the y_{sr} constitute a normal system of solutions of the adjoint system.

Note. If system (35) is regular, then the sum of characteristic numbers of the functions

$$e^{\int \Sigma p_{ss}\, dt} \quad \text{and} \quad e^{-\int \Sigma p_{ss}\, dt}$$

is equal to zero. But this condition is only necessary for regular systems as is shown by the example of the irregular system in section 65.

68. Example. Let us consider the properties of the variational equations for the canonical Hamiltonian equations (section 4)

$$\left.\begin{aligned} \frac{d\xi_j}{dt} &= \sum_i \left(\frac{\partial^2 H}{\partial p_j\, \partial q_i}\, \xi_i + \frac{\partial^2 H}{\partial p_j\, \partial p_i}\, \eta_i \right), \\ \frac{d\eta_j}{dt} &= -\sum_i \left(\frac{\partial^2 H}{\partial q_j\, \partial q_i}\, \xi_i + \frac{\partial^2 H}{\partial q_j\, \partial p_i}\, \eta_i \right), \end{aligned}\right\} \tag{38}$$

where the coefficients are continuous bounded real functions of t.

These equations are important for the study of stability of motion in conservative mechanical systems.

Poincaré established that if ξ_s, η_s and ξ'_s, η'_s are any two particular solutions of the variational equations (38), then

$$\sum_s (\xi_s \eta'_s - \eta_s \xi'_s) = C,$$

where C is some constant. The proof is elementary and is seen by differentiation with respect to t.

For each ξ_s, η_s there can always be found at least one other solution ξ'_s, η'_s, for which the constant C in the Poincaré invariant will be different from zero. Essentially, for a non-trivial solution ξ_s, η_s some of the quantities ξ_{s0}, η_{s0} of the initial magnitudes at the time t_0 will be different from zero; then the second particular solution may always be defined by the initial values ξ'_{s0}, η'_{s0} such that the constant of interest to us is different from zero.

For two solutions of the variational equations ξ_s, η_s and ξ'_s, η'_s let the value of the constant C be different from zero while λ and λ' are the characteristic numbers corresponding to these solutions. Using the Poincaré invariant, we derive the inequality

$$\lambda + \lambda' \leqslant 0.$$

Consequently, if the variational equations (38) have at least one particular solution with non-zero characteristic number, then there will exist at least one solution with negative characteristic number; the unperturbed motion will be unstable (in the first approximation).

The unperturbed motion may be stable only in the case where the characteristic numbers of solutions of systems (38) are zero.

The trace, i.e. Σp_{ss}, for equations (38) is always equal to zero; therefore, if the unperturbed motion is stable, the sum of characteristic numbers of any system of independent solutions of equations (38) is equal to the characteristic number of the expression

$$e^{-\int \Sigma p_{ss}\, dt}$$

taken with reversed sign. Consequently, for stable unperturbed motion the system of variational equations (38) will always be regular (section 67) and each complete system of independent solutions of equations (38) will be normal (sections 67 and 65).

Let us denote by

$$\xi_{sr},\ \eta_{sr} \qquad (r = 1, \ldots, 2n)$$

the system of independent solutions of equations (38), defined for $t = t_0$ by the values

$$\xi^0_{sr} = \delta_{sr},\ \eta^0_{sr} = \delta_{s,r-n}.$$

If the unperturbed motion is stable then the solutions ξ_{sr}, η_{sr} not only have zero characteristic numbers but are also bounded. Therefore the new system of variables

$$z_r = \sum_s (\xi_s \eta_{sr} - \eta_s \xi_{sr}) \qquad (r = 1, \ldots, 2n),$$

defined by a linear transformation with bounded coefficients (with the transformation determinant equal to +1 and having bounded minors) also has the same inverse transformation. The variables z_s satisfy the equations

$$\frac{dz_r}{dt} = 0 \qquad (r = 1, \ldots, 2n).$$

with constant coefficients.

Consequently, if the unperturbed motion is stable, the corresponding Poincaré variational equations (38) are reduced (using linear transformations with bounded coefficients having the same inverse transformation) to a system of equations with constant coefficients.

Let us investigate the group of transformations describing the perturbed motions of conservative systems.

The solutions ξ_s, η_s of the Poincaré variational equations (38), defined by the initial values ξ^0_s, η^0_s, are

$$\xi_s = \sum_{j=1}^{n} (\xi_j \xi_{s,j} + \eta^0_j \xi_{s,n+j}),$$

$$\eta_s = \sum_{j=1}^{n} (\xi^0_j \eta_{sj} + \eta^0_j \eta_{s,n+j})$$

$$(s = 1, \ldots, n).$$

The determinant of these linear transformations is

$$\Delta = 1.$$

This signifies that the relationships between ξ_s, η_s and ξ_s^0, η_s^0 are a group of unimodular linear transformations.

Buf if the unperturbed motion is stable then the variational equations (38), in their reduced form, should have a sign-definite quadratic form, the total derivative of which with respect to time is zero by reason of the variational equations.

On First-approximation Stability

69[12, 13L]. The characteristic numbers of the first-approximation equations permit us in many cases to make a decision on the stability and instability of the unperturbed motion by reason of the complete system of differential equations of perturbed motion

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n + X_s \qquad (s = 1, \ldots, n),$$

where the first-approximation coefficients p_{sr} and the coefficients of the functions X_s holomorphic with regard to $x_1, \ldots, x_n$ are real, continuous, bounded functions of t.

Lyapunov's Theorem. If the system of differential equations of the first-approximation is regular and if all its characteristic numbers are positive, then the unperturbed motion is stable.

Lyapunov proved this important theorem by using certain series, satisfying the equations of perturbed motion. But it is possible to prove it also by a direct method.

Let us consider the normal system of independent solutions $x_{1r}, \ldots, x_{nr} (r = 1, \ldots, n)$ for the first approximation equations. We construct the normal system of independent solutions of the adjoint system according to the formulae

$$y_{sr} = \frac{\Delta_{sr}}{\Delta},$$

where $\Delta = \|x_{sr}\|$, while Δ_{sr} denotes the minor of determinant Δ, corresponding to the element x_{sr} (section 67). Let $\lambda_1, \ldots, \lambda_n$ be the characteristic numbers of the solutions of these variational equations; then since the given system is assumed regular, the characteristic numbers of these

solutions of the adjoint system will be $-\lambda_1, \ldots, -\lambda_n$.

Let us introduce the new variables

$$z_r = \sum_s x_s y_{sr} e^{-(\lambda_r - \varepsilon)t} \qquad (r = 1, \ldots, n),$$

where ε signifies a certain positive number less than any of the characteristic numbers λ_r. Using the above transformation, by reason of the regularity of the variational equations, we find*:

$$\text{char. number of } \{z_r\} \geqslant \text{char. number of } \{x_s\} - \varepsilon;$$

the inverse transformation

$$\sum_r z_r e^{(\lambda_r - \varepsilon)t} x_{sr} = \sum_s x_s \left(\sum_r y_{sr} x_{sr}\right) = x_s$$

leads to the inequality

$$\text{char. number of } \{x_s\} \geqslant \text{char. number of } \{z_r\} + \varepsilon.$$

Consequently,

$$\text{char. number of } \{z_r\} = \text{char. number of } \{x_s\} - \varepsilon.$$

Let us consider the quadratic form,

$$2V = \sum_r z_r^2.$$

sign-definite with respect to $z_1, \ldots, z_n$. Its total derivative with respect to t is

$$\frac{dV}{dt} = -\sum_r (\lambda_r - \varepsilon) z_r^2 + R,$$

where

$$R = \sum_{r,s} z_r X_s y_{sr} e^{-(\lambda_r - \varepsilon)t};$$

as a function of the new variables $R(t, z_1, \ldots, z_n)$ has in its expansion in positive integral powers of the variables $z_1, \ldots, z_n$ coefficients which are vanishing functions of t for each positive ε; the expansion of R begins with terms of at least the third degree.

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* The symbol $\{z_r\}$ denotes the system of functions of $z_1, \ldots, z_n$.

For numerically small values of the variables z_r the derivative $\frac{dV}{dt}$ will be a negative-definite function of the quantities $z_1, \ldots, z_n$. By virtue of Lyapunov's theorem on instability we conclude from this that the unperturbed motion is stable with respect to the variables $z_1, \ldots, z_n$; here each perturbed motion sufficiently close to the unperturbed tends to the latter asymptotically with respect to z_r.

As the function $R(t, z_1, \ldots, z_n)$ has vanishing coefficients and begins its expansion with terms of at least the third order with regard to z_s, it is possible to find for each arbitrarily small positive η, and for each t greater than some constant T, a region of sufficiently small numerical values of $z_1, \ldots, z_n$, within which

$$|R(t, z_1, \ldots, z_n)| < \eta \sum_r z_r^2.$$

Under these conditions, during the entire time $t > T$, as long as the values of variables $z_1, \ldots, z_n$ do not leave this region, if λ_1 is the smallest of the characteristic numbers $\lambda_1, \ldots, \lambda_n$, we have

$$\frac{d}{dt} \cdot \frac{1}{2} \sum_r z_r^2 \leqslant -(\lambda_1 - \varepsilon - \eta) \sum_r z_r^2,$$

and consequently, if the initial values z_{s0} are taken such that as t varies from t_0 to T the values of the variables are located in these regions while the constant η satisfies the inequality $\lambda_1 > \varepsilon + \eta$, with further variation of t we will have

$$\sum_r z_r^2 \leqslant Ce^{-2(\lambda_1 - \varepsilon - \eta)t}.$$

From this

$$\text{char. number of } \{z_r\} \geqslant \lambda_1 - \varepsilon - \eta$$

and, consequently,

$$\text{char. number of } \{x_s\} \geqslant \lambda_1 - \eta > 0.$$

This proves the stability of the unperturbed motion with regard to the variables $x_1, \ldots, x_n$ and also shows that each sufficiently close perturbed motion tends to it asymptotically.

70. Theorem. If the system of differential equations of the first approximation is regular and among its characteristic numbers at least one is negative, the unperturbed motion is unstable.

Proof. Let us introduce the new variables

$$z_r = \sum_s x_s y_{sr} e^{-\lambda_r t}.$$

From this we see that the characteristic number of the function z_r is not less than the smallest of characteristic numbers of the functions $x_s y_{sr} e^{-\lambda_r t}$, and the latter is not less than the characteristic number of the system of functions $x_1, \ldots, x_n$, since the characteristic number $y_{sr} e^{-\lambda_r t}$ is not less than zero, if the first-approximation equation is regular. This relationship is valid for each r; therefore

$$\text{char. number } \{z_r\} \geqslant \text{char. number } \{x_s\}.$$

For the variables z_r we have the following differential equations:

$$\frac{dz_r}{dt} = -\lambda_r z_r + \sum_s X_s y_{sr} e^{-\lambda_r t} \qquad (r = 1, \ldots, n).$$

Let us assume that among the characteristic numbers $\lambda_1, \ldots, \lambda_n$ there is at least one, say λ_1, negative. The instability of the unperturbed motion (with regard to the variables $x_1, \ldots, x_n$) will be proved by reductio ad absurdum.

If the unperturbed motion is stable, then for a given small positive number A there will exist such a positive number R that for arbitrary initial perturbations $x_{10}, \ldots, x_{n0}$, satisfying the inequality

$$\sum_s x_{s0}^2 < R, \tag{39}$$

for each t greater than t_0, the inequality

$$\sum_s x_s^2 < A. \tag{40}$$

will be satisfied.

From the equation $r = 1$ of the last system of equations it follows that

$$z_1 = ce^{-\lambda_1 t} + e^{-\lambda_1 t} \int \sum_s X_s y_{s1}\, dt,$$

where c is some constant. The expansion of the functions X_s in positive integral powers of $x_1, \ldots, x_n$ begins with a term of second order; the functions X_s are assumed bounded for each t greater than t_0 and for each $x_1, \ldots, x_n$, satisfying condition (40); therefore with choice of initial values $x_{10}, \ldots, x_{n0}$ in accordance with inequality (39) for sufficiently small R we conclude that if the unperturbed motion is stable, the characteristic number of the second term of the right part of the last equality, by reason of the proposition on the characteristic number of an integral (section 62), is not less than zero. Consequently the characteristic number of the function z_1 will be equal to λ_1, since, with the original assumptions, the constant c is different from zero.

But the characteristic number of $\{x_s\} \leqslant$ characteristic number of $\{z_r\}$; therefore

$$\text{char. number of } \{x_s\} \leqslant \lambda_1 < 0.$$

This inequality is incompatible with condition (40). We must therefore conclude that whatever the value of R, constraining the choice of initial perturbation $x_{10}, \ldots, x_{n0}$, among the latter there are found some values for which inequality (40) ceases to be satisfied for certain values of t greater than t_0.

Thus the theorem may be considered proved.

Note. If the system of differential equations of the first approximation is not regular, then, denoting by S the sum of all characteristic numbers of the normal system of its solutions, and by μ the characteristic number of the function $\frac{1}{\Delta}$, we will have

$$S + \mu = -\sigma,$$

where σ is a certain positive number. In this case the characteristic number of the function $\frac{\Delta_{ij}}{\Delta}$ is not less than $-\lambda_j - \sigma$. But on the basis of this it is not difficult to prove (by considerations similar to the above and those in section 69) the following propositions:

Let the system of differential equations of the first

approximation be irregular; if it has all its characteristic numbers greater than σ, then the unperturbed motion is stable and if, among its characteristic numbers is found at least one negative, and numerically greater than σ, the unperturbed motion is unstable*.

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* Other criteria of stability in the first approximation were presented in 1, 2, 3.

CHAPTER 10

PERIODIC MOTIONS

The Invariant Substitution and the Form of Particular Solutions

71[46L]. Consider the system of linear differential equations

$$\frac{dx_s}{dt} = p_{s1}x_1 + \ldots + p_{sn}x_n \qquad (s = 1, \ldots, n) \tag{41}$$

with the assumption that all the coefficients p_{sr} are real, periodic, bounded, continuous functions of the real variable t with a common real period ω $(\omega > 0)$.

In many practical cases the variational equations for a forced periodic motion are differential equations of this type.

Problems about the stability of the unperturbed motion $(x_1 = 0, \ldots, x_n = 0)$ lead to the determination of the characteristic numbers for such equations. Unfortunately, this last question has not been completely solved, although its solution is facilitated by the existence of the invariant substitution

$$S = \begin{pmatrix} t, & x_1, \ldots, x_n \\ t+\omega, & x_1, \ldots, x_n \end{pmatrix},$$

which does not change the form of the given equations (41).

The invariant substitution, being applied an unlimited number of times to the values of t on the segment $(0, \omega)$, covers the entire range of the real variable t. In other words, from the behaviour of the general solutions x_s with t varying in the segment $(0, \omega)$ it is possible to determine the behaviour of each particular solution x_s with unbounded variation of t.

Let us assume that for equations (41) there have been

found n independent solutions

$$x_{1r}, \ldots, x_{nr} \qquad (r = 1, \ldots, n).$$

The functions

$$x_{1r}(t+\omega), \ldots, x_{nr}(t+\omega)$$

from the property of periodic equations, with the invariant substitution S, also represent their solution. Therefore they will be linearly expressed through independent solutions x_{sr}:

$$x_{sr}(t+\omega) = a_{1r}x_{s1}(t) + \ldots + a_{nr}x_{sn}(t), \quad (s = 1, \ldots, n),$$

where a_{kr} are certain constants for each r chosen from the numbers $1, \ldots, n$.

From the independent solutions chosen by us we may pass to another similar system by linear transformations with constant coefficients; in particular we may pass to a certain solution

$$x_s = \beta_1 x_{s1} + \ldots + \beta_n x_{sn},$$

having the fundamental property

$$x_s(t+\omega) = \varkappa x_s(t) \qquad (s = 1, \ldots, n).$$

To determine the constants β_r from this we have the relationships

$$\begin{aligned}\beta_1 [a_{11}x_{s1}(t) + \ldots + a_{n1}x_{sn}(t)] + \ldots + \\ + \beta_n [a_{1n}x_{s1}(t) + \ldots + a_{nn}x_{sn}(t)] = \\ = \varkappa [\beta_1 x_{s1}(t) + \ldots + \beta_n x_{sn}(t)].\end{aligned}$$

These relationships $(s = 1, \ldots, n)$ should be satisfied independently of the value of t; therefore, by reason of the properties of linearly independent systems of solutions, the coefficients for identical x_{sr} can be equated:

$$\begin{aligned}a_{11}\beta_1 + \ldots + a_{1n}\beta_n &= \varkappa\beta_1, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}\beta_1 + \ldots + a_{nn}\beta_n &= \varkappa\beta_n.\end{aligned}$$

This system of linear homogeneous algebraic expressions with constant coefficients will admit a non-trivial solution for constants β_r, if the determinant formed from

its coefficients satisfies the equation

$$\chi(\varkappa) = \|a_{ij} - \delta_{ij}\varkappa\| = 0. \tag{42}$$

The roots $\varkappa_1, \ldots, \varkappa_n$ of this determinant $\chi(\varkappa)$ are connected with the fundamental particular solutions of our equations.

72[46L]. Let us assume that the roots $\varkappa_s$ are all distinct. Each root is associated with one of the expressions of the n fundamental independent solutions. Let $\varkappa$ be any of these roots. By definition the corresponding solution has the property

$$x_s(t+\omega) = \varkappa x_s(t) \qquad (s = 1, \ldots, n).$$

The general form of the continuous, single-valued functions $x_s(t)$, bounded on $(0, \omega)$, satisfying the last relationship, is given by the formulae

$$x_s(t) = e^{\frac{t}{\omega}\ln\varkappa}\varphi_s(t),$$

where φ_s are continuous bounded single-valued functions with period ω. If the roots $\varkappa_r$ are all distinct, the n fundamental solutions obtained will form a system of n independent solutions having, obviously, the properties of a normal system.

When there exist multiple roots $\varkappa_r$, it is not always possible to find n independent solutions of this fundamental form. In this case it is possible to find a fundamental system of independent solutions defined for each of the distinct roots $\chi(\varkappa)$ by the relationships

$$\left.\begin{aligned} x_s^{(1)}(t+\omega) &= \varkappa x_s^{(1)}(t), \\ x_s^{(2)}(t+\omega) &= \varkappa x_s^{(2)}(t) + x_s^{(1)}(t), \\ &\cdots\cdots\cdots\cdots \\ x_s^{(m+1)}(t+\omega) &= \varkappa x_s^{(m+1)}(t) + x_s^{(m)}(t) \\ &(s = 1, \ldots, n), \end{aligned}\right\} \tag{43}$$

where $x_1^{(i)}, \ldots, x_n^{(i)}$ denote the required fundamental solutions connected with the multiple root $\varkappa$.

Let

$$x_s^{(i)} = \beta_1^{(i)}x_{s1} + \ldots + \beta_n^{(i)}x_{sn}.$$

To determine the constants $\beta_r^{(i)}$ and the highest value

for m after substitution of these expressions in the preceding relationships and after equating coefficients for identical functions of $x_{sr}(t)$ we have the following systems of equations ($i = 1, \ldots, n$):

$$\sum_j (a_{ij} - \delta_{ij}\varkappa)\,\beta_j^{(1)} = 0,$$
$$\sum_j (a_{ij} - \delta_{ij}\varkappa)\,\beta_j^{(2)} = \beta_i^{(1)},$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$\sum_j (a_{ij} - \delta_{ij}\varkappa)\,\beta_j^{(m+1)} = \beta_i^{(m)}.$$

These systems of equations coincide with system (7), studied in detail in sections 24-28. We give here the results obtained for such systems.

Let all the elementary divisors of the determinant $\chi(\varkappa)$ be of the form

$$(\varkappa - \varkappa_1)^{n_1}, \ldots, (\varkappa - \varkappa_k)^{n_k},$$

where $\varkappa_1, \ldots, \varkappa_k$ denote the roots of the determinant $\chi(\varkappa)$; among them may be identical roots. For each such root $\varkappa = \varkappa_r$ ($r = 1, \ldots, k$) the number m of the corresponding group (43) of fundamental solutions $x_1^{(i)}, \ldots, x_n^{(i)}$ will be:

$$m = n_r - 1.$$

The sum

$$n_1 + \ldots + n_k = n.$$

The formulae for determining the constants $\beta_s^{(i)}$ are of no interest since the constants a_{ij} and the particular solutions x_{sr} are known to us so far only from the existence theorem. The form of the functions $x_s^{(j)}$ entering into the fundamental system of equations (corresponding to the root $\varkappa$) is determined by the relationship (43).

From the system (43), we can determine the form of functions $x_1^{(j)}, \ldots, x_n^{(j)}$ for successive values of j; thus

$$x_s^{(1)}(t) = e^{\frac{t}{\omega}\ln x}\varphi_s^{(1)}(t),$$

$$x_s^{(2)}(t) = e^{\frac{t}{\omega}\ln x}\left[\varphi_s^{(1)}(t)\frac{t-\omega}{x\omega} + \varphi_s^{(2)}(t)\right],$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$x_s^{(m+1)}(t) = e^{\frac{t}{\omega}\ln x}\left[\varphi_s^{(1)}(t)\frac{(t-\omega)\ldots(t-m\omega)}{x^m\omega^m\, m!} + \right.$$

$$\left. + \varphi_s^{(2)}(t)\frac{(t-\omega)\ldots(t-(m-1)\omega)}{x^{m-1}\omega^{m-1}(m-1)!} + \ldots + \varphi_s^{(m+1)}(t)\right],$$

where all functions $\varphi_s^{(j)}$ are bounded and periodic with period ω. It is possible to verify this directly by substituting $t+\omega$ in place of t and also by successive integration of the finite-difference equations (43).

From the form of the fundamental solutions, we note that the real parts of the quantities $-\ln x_r/\omega$, $(r=1,\ldots,k)$ form a set of $n_1+\ldots+n_k=n$ characteristic numbers of the system of equations (41). The complete system of fundamental solutions $x_1^{(j)}, \ldots, x_n^{(j)}$ forms a normal system of $n_1+\ldots+n_k=n$ independent solutions, since each linear combination of these solutions with constant coefficients has a characteristic number which is obviously the smallest of the characteristic numbers entering into the combination of fundamental solutions.

That the system of independent solutions found is normal may be proved also in the following manner. By reason of relationships (43), the determinant $\Delta = \|x_{sr}\|$, composed of the system of fundamental solutions, satisfies the equality

$$\Delta(t+\omega) = x_1^{n_1}\ldots x_k^{n_k}\Delta(t);$$

we can verify this directly by operating on the columns. But from Liouville's formula (section 65) we have:

$$\Delta(t+\omega) = \Delta(t)\, e^{\int_0^\omega \Sigma p_{ss}\, dt}.$$

Comparing this with the preceding formula we find

$$x_1^{n_1}\ldots x_k^{n_k} = e^{\int_0^\omega \Sigma p_{ss}\, dt}.$$

Taking the logarithm of both sides of this equation, we see, after division by ω, that the sum of the characteristic numbers of the fundamental system of independent solutions is equal to

$$-\frac{1}{\omega}\int_0^{\omega}\sum p_{ss}\,dt;$$

now this quantity is the characteristic number of the function

$$\int_{e^0}^{I}\Sigma\, p_{ss}\,dt$$

since the expression

$$\int_0^t\sum p_{ss}\,dt-\frac{t}{\omega}\int_0^{\omega}\sum p_{ss}\,dt$$

represents a bounded, periodic function (with period ω). From this we conclude that the fundamental system of independent solutions will be normal and that the given system of differential equations with periodic coefficients is regular (sections 64, 65, 67).

73[47L]. Let us consider the system of equations

$$\frac{dy_s}{dt}+p_{1s}y_1+\ldots+p_{ns}y_n=0,$$

adjoint to the given one. It also has periodic coefficients with real period ω; thus it is also possible to determine the fundamental system of independent solutions, divided into groups corresponding to the elementary divisors

$$(\mu-\mu_r)^{n_r}\qquad(r=1,\ldots,k)$$

of the determinant $\chi(\mu)$ of the adjoint system.

Let the solution of the group of a given elementary divisor be

$$y_s=e^{\frac{t}{\omega}\ln\mu_r}\left[\psi_s^{(1)}\frac{(t-\omega)\ldots(t-(n_r-1)\omega)}{\mu_r^{(n_r-1)}\omega^{n_r-1}(n_r-1)!}+\ldots+\psi_s^{(n_s)}\right],$$

where $\psi_s^{(j)}$ are functions which are bounded and periodic with period ω.

Substituting this solution in the coefficients of the linear integral of the given equations (41)

$$y_1x_1+\ldots+y_nx_n=c,$$

we obtain, after an obvious transformation, the expression

$$\left[z_1^{(r)}\frac{t^{n_r-1}}{(n_r-1)!}+\ldots+z_n^{(r)}\right]e^{\frac{t}{\omega}\ln\mu_r}=c,$$

where all the $z_j^{(r)}$ denote certain linear forms of the quantities x_s with periodic coefficients; c is a constant. Letting r vary from 1 to k, we obtain n independent linear forms $z_j^{(r)}$ which may be taken as the new variables in place of $x_1, \ldots, x_n$.

Differentiating the last expression with respect to t and equating the coefficients of identical powers of t to zero, we obtain:

$$\begin{aligned}&\frac{dz_1^{(r)}}{dt}=\lambda_r z_1^{(r)},\\&\frac{dz_j^{(r)}}{dt}=\lambda_r z_j^{(r)}-z_{j-1}^{(r)}\end{aligned}\qquad\begin{pmatrix}j=2,\ \ldots.\ n_r\\r=1,\ \ldots.\ k\end{pmatrix},$$

where

$$\lambda_r=-\frac{1}{\omega}\ln\mu_r.$$

Consequently, the system (41) is transformed into a system with constant coefficients; here the transformation has been carried out by means of a non-singular linear substitution with periodic bounded coefficients, which does not change the characteristic numbers of the normal system of independent solutions. From this we conclude that

$$\ln\varkappa_r+\ln\mu_r=0,$$

i.e. the characteristic numbers of the equations of the adjoint system are equal to the characteristic numbers of the given system with reversed sign, while the powers of the elementary divisors of the roots $\varkappa_r$ and μ_r are equal to each other*.

From the dependence of the characteristic numbers on the roots of the characteristic equation (42), $\chi(\varkappa)=0$, and the theorems on stability and instability for regular systems (sections 69, 70), we derive the theorem:

---

* General studies on the applicability of linear equations to equations with constant coefficients, begun by Lyapunov [10L], were developed further by Erugin[1].

When the characteristic equation $\gamma(\varkappa) = 0$ has only roots with moduli less than one, the unperturbed motion is stable, independently of terms higher than the first order in X_s, and such that each perturbed motion sufficiently close to the unperturbed approaches it asymptotically. When among the roots of the equation are found some with moduli greater than one, this motion is unstable independently of X_s.

Therefore with regard to the question of stability in the presence of terms X_s in the differential equations of perturbed motion, the case which remains undecided is that in which the characteristic polynomial $\gamma(\varkappa)$ (not having roots with moduli greater than one) has roots with moduli equal to one.

Approximate Methods Of Determining The Characteristic Equation

74. The determination of the characteristic equation for linear differential equations with periodic coefficients is a difficult problem. Although for equations with constant coefficients the formation of the characteristic determinant $\Delta(\lambda)$ does not require knowledge of the particular solutions, in the case of equations with periodic coefficients, in the present state of the subject, such knowledge is necessary in order to form the equation $\gamma(\varkappa) = 0$. Only the constant term of this equation may be found directly from the formula

$$\varkappa_1 \ldots \varkappa_n = e^{\int_0^\omega \Sigma p_{ss}\, dt}.$$

To determine the characteristic equation $\gamma(\varkappa) = 0$ one of the approximate methods of integrating the differential equations is usually used. The most convenient solutions for this purpose are those independent particular solutions of equations (41) which are defined for $t = 0$ by the following initial values:

$$x_{sr}(0) = \delta_{sr}.$$

The elements a_{sr} of the characteristic determinant $\gamma(\varkappa)$ are calculated then by the equations (section 71):

$$a_{sr} = x_{sr}(\omega),$$

and this means that

$$\chi(\varkappa) = \| x_{sr}(\omega) - \delta_{sr}\varkappa \|.$$

In essence, stability problems are simpler than problems of integration. The exact knowledge of the characteristic numbers is not required; it is necessary to know only their signs. But to solve such a problem, methods may be used which, in integration problems, lead to divergent processes. This idea was first applied by Lyapunov in studying the stability of steady-state motion in critical cases. Naturally, each particular result obtained by such procedures should be tested by rigorous theorems on stability or instability.

75. Let us consider a procedure which consists of replacing the coefficients p_{sr} in the given equations by their mean values over the period

$$c_{sr}^{(1)} = \frac{1}{\omega}\int_0^{\omega} p_{sr}\,dt.$$

The stability (instability) of the averaged system

$$\frac{dx_s}{dt} = c_{s1}^{(1)}x_1 + \ldots + c_{sn}^{(1)}x_n \qquad (s = 1, \ldots, n)$$

and that of the unperturbed motion which satisfies the given equations (41) sometimes go together, but this is not always true. It is therefore important to know when such a substitution permits the given stability problem to be solved.

We shall calculate the independent solutions x_{sr} of the system (41), defined by the initial values $x_{sr}^{(0)} = x_{sr}(0) = \delta_{sr}$, by Picard's well-known method of successive approximations:

$$x_{sr}^{(k)} = \delta_{sr} + \int_0^t \sum_i p_{si}x_{ir}^{(k-1)}\,dt \qquad (k = 1, 2, \ldots).$$

Here the characteristic equation of system (41)

$$\chi(\varkappa) = \| x_{sr}(\omega) - \delta_{sr}\varkappa \| = 0$$

is defined in the k-th approximation as

$$\chi_k(\varkappa) = \| x_{sr}^{(k)}(\omega) - \delta_{sr}\varkappa \| = 0.$$

In the k-th approximation we may relate the equations with constant coefficients

$$\frac{dx_s}{dt} = c_{s1}^{(k)} x_1 + \ldots + c_{sn}^{(k)} x_n, \tag{44}$$

to the system (41) where

$$c_{sr}^{(k)} = \frac{1}{\omega} \int_0^{\omega} \sum_i p_{si} x_{ir}^{(k-1)}\, dt.$$

The roots $\lambda_1^{(k)}, \ldots, \lambda_n^{(k)}$ of the characteristic equation of system (44)

$$\Delta_k(\lambda) = \| c_{sr}^{(k)} - \delta_{sr}\lambda \| = 0$$

are related to the roots $\varkappa_1^{(k)}, \ldots, \varkappa_n^{(k)}$ of the equations $\gamma_k(\varkappa) = 0$ by the relations

$$\varkappa_s^{(k)} - 1 = \omega \lambda_s^{(k)}.$$

If for whole non-negative values $m_1, \ldots, m_n$ whose sum is 2, the expression

$$m_1\lambda_1 + \ldots + m_n\lambda_n$$

does not vanish, where λ_s are roots of $\Delta_k(\lambda)$, then, in accordance with the discussion in section 34, it is possible to find the real quadratic form with symmetrical coefficients $b_{rs} = b_{sr}$

$$2V_k = \sum_{rs} b_{sr} x_s x_r$$

such that

$$\sum_s (c_{s1}^{(k)} x_1 + \ldots + c_{sn}^{(k)} x_n) \frac{\partial V_k}{\partial x_s} = x_1^2 + \ldots + x_n^2.$$

The total derivative of V_k with respect to t is, from the given equations (41),

$$V_k' = \sum_{r,s,i} b_{si} p_{ir} x_s x_r.$$

This derivative V_k' will be positive-definite if there is a positive number μ such that the real quadratic form with periodic coefficients

$$V_k' - \mu(x_1^2 + \ldots + x_n^2) = \sum_{irs} b_{si}(p_{ir} - \mu c_{ir}^{(k)}) x_s x_r$$

has the discriminant

$$\| k_{rs}^{(k)} \|,$$

the principal diagonal minors of which will all be positive, where

$$h_{rs}^{(k)} = \frac{1}{2}\sum_i [b_{ri}(p_{is} - \mu c_{is}^{(k)}) + b_{si}(p_{ir} - \mu c_{ir}^{(k)})] = h_{sr}^{(k)}.$$

But if V_k' is positive-definite, the asymptotic stability (instability) of the unperturbed motion in equations (44) accompanies such stability (instability) in the given equations (41).

In the first approximation ($k = 1$), system (44) coincides with the averaged system, and the above analysis gives the sufficient conditions (in the form of the positive-definiteness of V_1') for the asymptotic stability (instability) of the unperturbed motion in the averaged equations ($k = 1$) to accompany asymptotic stability (instability) of the unperturbed motion in the given equations (41).

Note. If in any of the cases considered the roots λ_s are such that there exist non-negative integers m_s whose sum is 2, for which the expression $m_1\lambda_1 + \ldots + m_n\lambda_n$ vanishes and there exists at least one negative characteristic number λ_s, then by introduction of the new variables

$$u_s = x_s e^{-\eta t}$$

we find that for sufficiently small positive η_i the problem reduces to the case considered, since the variations of variables x_s in the course of time will not all be bounded, if certain of the variables u_j are unbounded. The determination of the independent solutions x_{sr} by Picard's method for the zero approximation

$$x_{sr}^{(0)} = \delta_{sr}$$

leads generally to a rather slow convergence of the successive approximations, since it is necessary to carry out a sufficient number of approximations to accumulate a sufficient number of terms of the exponential $e^{\alpha t}$ in increasing powers of t, which in this process cannot appear except as a series expansion.

Example. Let us consider the system

$$\frac{dx}{dt} = \left(-\frac{1}{2} + \varepsilon a\cos 2t\right)x + (1 - \varepsilon a\sin 2t)\,y,$$

$$\frac{dy}{dt} = (-1 - \varepsilon a\sin 2t)\,x + \left(-\frac{1}{2} - \varepsilon a\cos 2t\right)y,$$

where a is some positive number and ε is a parameter. Their averaged equations are

$$\frac{dx}{dt} = -\frac{1}{2}x + y,$$
$$\frac{dy}{dt} = -x - \frac{1}{2}y.$$

For these equations the function V just discussed represents a negative definite quadratic form

$$V = -(x^2 + y^2).$$

In other words, the unperturbed motion $(x = 0,\ y = 0)$ is asymptotically stable in the system of equations with averaged coefficients. The total derivative with regard to t of this function V is, from the given equations,

$$V' = x^2 + y^2 - 2\varepsilon a\,[(x^2 - y^2)\cos 2t - 2xy \sin 2t];$$

the discriminant of V' has all its principal diagonal minors positive if the numerical value of ε is less than $\frac{1}{2a}$. Consequently, if $|\varepsilon| < \frac{1}{2a}$, the unperturbed motion will be asymptotically stable.

It may be noted that for $\varepsilon = \frac{3}{2a}$ the given equations make the unperturbed motion unstable.

Lyapunov's Method

76[48L]. Lyapunov developed another method of calculating independent solutions x_{sr}, defined by the initial values $x_{sr}^{(0)} = \delta_{sr}$.

Let us consider the case where, in system (41), the coefficients p_{sr} depend on some real parameter μ. Let us assume that the coefficients p_{sr} may be represented by series in positive integral powers of μ, which converge absolutely and uniformly for all real values of t, at least for $|\mu|$ not exceeding some number M. We shall also assume that as long as $|\mu|$ does not exceed M the coefficients p_{sr} are continuous, bounded functions for each $t > 0$; let the period ω be independent of μ.

Let p be a series, expanded in positive integral powers of μ, with coefficients which are periodic functions of t; the values of these coefficients for any power of μ and for

each $t > 0$ are to equal the largest modulus among the coefficients of the expansions of p_{sr}, corresponding to the same power of μ and the same value of t.

If for

$$|\mu| = M$$

the series p converges uniformly for all real values of t. the series represented by the values

$$x_{sr}(\omega) \qquad (s,\ r = 1,\ \ldots,\ n)$$

of the functions x_{sr} of the system of independent solutions, defined by the initial conditions $x_{sr}(0) = x_{sr}^{(0)} = \delta_{sr}$, will under these assumptions be absolutely convergent for $|\mu| = M$.

Proof. Consider the system

$$\frac{dx_s}{dt} = \varepsilon(p_{s1}x_1 + \ldots + p_{sn}x_n)$$

with a new auxiliary parameter ε. The solutions x_{sr} for the last equations will be sought in the form

$$x_{sr} = \delta_{sr} + \varepsilon x_{sr}^{(1)} + \varepsilon^2 x_{sr}^{(2)} + \ldots;$$

the functions $x_{sr}^{(k)}$ are determined recursively from the formula

$$x_{sr}^{(k)} = \int_0^t \sum_j p_{sj} x_{jr}^{(k-1)}\, dt \qquad (k = 1,\ 2,\ \ldots),$$

from which we see that

$$|x_{sk}^{(1)}| \leqslant n \int_0^t p\, dt,$$

$$|x_{sk}^{(2)}| \leqslant n^2 \int_0^t \left(p \int_0^t p\, dt\right) dt.$$

Integrating by parts the expression on the right-hand side of the last inequality, we obtain

$$|x_{sr}^{(2)}| \leqslant \frac{n^2}{2}\left(\int_0^t p\, dt\right)^2,$$

and analogously

$$|x_{sr}^{(k)}| \leqslant \frac{n^k}{1 \cdot 2 \ldots k} \left(\int_0^t p\, dt \right)^k.$$

Consequently, for $\varepsilon > 0$ and $t \geqslant 0$ we have:

$$|\varepsilon x_{sr}^{(1)}| + |\varepsilon^2 x_{sr}^{(2)}| + \ldots \leqslant e^{|\varepsilon| n \int_0^t p\, dt} - 1.$$

For $\varepsilon = 1$ and $t = \omega$, under the given assumptions, we conclude (from the last inequality) that $x_{sr}(\omega)$ can be expanded in an absolutely convergent series in powers of μ for $|\mu| = M$. The proposition is proved.

We shall now assume that we can integrate the system (41) for $\mu = 0$. Then, if the functions x_s for $\mu \neq 0$ are sought in the form of series, expanded in powers of the parameter μ, we obtain a system of differential equations to determine the coefficients; these equations can be integrated in a definite sequence by means of quadratures.

77[49L]. Let us consider the equation

$$\frac{d^2x}{dt^2} + px = 0, \tag{45}$$

where p is a bounded, continuous, periodic function of t with real (say positive) period ω. In accordance with the above method (using the small parameter μ) we change this equation to

$$\frac{d^2x}{dt^2} = \mu p x;$$

we shall look for independent solutions of the last equation in the form

$$x = f = 1 + \mu f_1 + \mu^2 f_2 + \ldots$$
$$x = \varphi = t + \mu\varphi_1 + \mu^2\varphi_2 + \ldots,$$

where f_n, φ_n are functions of t which vanish, together with their first derivatives, when t is equal to zero. The functions f_n, φ_n are calculated from the formulae

$$f_n = \int_0^t dt \int_0^t p f_{n-1}\, dt, \quad \varphi_n = \int_0^t dt \int_0^t p\varphi_{n-1}\, dt$$

with the condition

$$f_0 = 1, \quad \varphi_0 = t.$$

If we replace the equation by the system of linear equations

$$\frac{dx}{dt} = x', \quad \frac{dx'}{dt} = \mu p x,$$

we find (section 71) that the characteristic equation is

$$\chi(\varkappa) = \begin{vmatrix} f(\omega) - \varkappa & f'(\omega) \\ \varphi(\omega) & \varphi'(\omega) - \varkappa \end{vmatrix} = \varkappa^2 - 2A\varkappa + 1 = 0,$$

where

$$2A = f(\omega) + \varphi'(\omega).$$

If $A^2 > 1$, the roots of the characteristic equation will be real, one of them being numerically greater, the other numerically smaller than 1; if $A^2 \leqslant 1$, the characteristic roots will be complex and will have moduli equal to 1.

From the formula

$$A = 1 + \frac{1}{2}\sum_{n=1}^{\infty} [f_n(\omega) + \varphi_n'(\omega)]\,\mu^n$$

and from the fact that, for $p \leqslant 0$, the functions $f_n(\omega)$, $\varphi_n'(\omega)$ are positive for even n and negative for odd n, we deduce that if the function p only takes on negative or zero values (not vanishing identically), the roots of the characteristic equation corresponding to the given equation ($\mu = -1$) will always be real and one of them in modulus greater, the other less, than 1.

Let us now consider the case where the function p may have only non-negative values, not vanishing identically.

Integrating by parts and using the relations $f_0 = 1$, $\varphi_0 = t$, we obtain the equation

$$f_1(\omega) + \varphi_1'(\omega) = \omega \int_0^{\omega} p\,dt.$$

For positive t and for $n > 1$ the functions $f_n(t)$, $\varphi_n'(t)$ satisfy the inequality

$$S_n = (f_{n-1} + \varphi_{n-1}')\,t \int_0^t p\,dt - 2n(f_n + \varphi_n') > 0.$$

Now from the formula

$$S_n = \int_0^t \frac{dS_n}{dt}\, dt$$

we have directly:

$$S_n = \int_0^t (F_n + p\Phi_n)\, dt,$$

where

$$F_n = tf'_{n-1}\int_0^t p\, dt + (f_{n-1} + \varphi'_{n-1})\int_0^t p\, dt - 2nf'_n,$$

$$\Phi_n = t\varphi_{n-2}\int_0^t p\, dt + (f_{n-1} + \varphi'_{n-1})\, t - 2n\varphi_{n-1}.$$

Carrying out on F_n, Φ_n a similar transformation to that carried out on the function S_n, we obtain

$$F_n = \int_0^t \left(2f'_{n-1}\int_0^t p\, dt + pu_n\right) dt, \quad \Phi_n = \int_0^t (2pt\varphi_{n-2} + v_n)\, dt,$$

where

$$u_n = (\varphi_{n-2} + tf_{n-2})\int_0^t p\, dt + \varphi'_{n-1} + tf'_{n-1} - (2n-1)\, f_{n-1},$$

$$v_n = (\varphi_{n-2} + t\varphi'_{n-2})\int_0^t p\, dt + f_{n-1} + tf'_{n-1} - (2n-1)\, \varphi'_{n-1}.$$

With the assumption that p is not negative, the functions f_n, f'_n, φ_n, φ'_n for each n will be positive for $t > 0$. And, consequently, the functions F_n, Φ_n will be positive if u_n, v_n are positive. But if we carry out, on functions u_n, v_n, the transformation applied to S_n, Φ_n, F_n, it is possible to obtain:

$$u_n = \int_0^t (2p\,(\varphi_{n-2} + tf_{n-2}) + F_{n-1})\, dt,$$

$$v_n = \int_0^t (2f'_{n-1} + 2\varphi'_{n-2}\int_0^t p\, dt + p\Phi_{n-1})\, dt.$$

From these formulae we conclude that if, for positive values of t the inequalities $F_{n-1} > 0$, $\Phi_{n-1} > 0$ are valid, then, for such values of t, the inequalities $F_n > 0$, $\Phi_n > 0$ are valid, and thus the $S_n > 0$.

For $n=2$ and positive t we have:

$$F_2 = 2\int_0^t \left\{\left(\int_0^t p\,dt\right)^2 + 2p\varphi_1'\right\} dt > 0, \quad \Phi_2 = 2\int_0^t (pt^2+2f_1)\,dt > 0.$$

from this we can prove that $S_n > 0$. Putting $t=\omega$, we then obtain:

$$f_n(\omega)+\varphi_n'(\omega) < [f_{n-1}(\omega)+\varphi_{n-1}'(\omega)]\frac{\omega}{2n}\int_0^\omega p\,dt.$$

The expression A for equation (45) takes the form

$$A = 1+\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n[f_n(\omega)+\varphi_n'(\omega)].$$

From the last inequality we find:

$$1-\frac{\omega}{2}\int_0^\omega p\,dt+\frac{1}{2}\sum_{n=1}^{\infty}\left(1-\frac{\omega}{4n+2}\int_0^\omega p\,dt\right)[f_{2n}(\omega)+\varphi_{2n}'(\omega)] < A,$$

$$A < 1-\frac{1}{2}\sum_{n=1}^{\infty}\left(1-\frac{\omega}{4n}\int_0^\omega p\,dt\right)[f_{2n-1}(\omega)+\varphi_{2n-1}'(\omega)].$$

Consequently, if

$$\omega\int_0^\omega p\,dt \leqslant 4,$$

we have necessarily

$$-1 < A < 1;$$

and, from this, the following theorem of Lyapunov is proved.

If the function p is such that it may take on only positive or zero values (not being zero identically), and if the function satisfies the condition

$$\omega\int_0^\omega p\,dt \leqslant 4,$$

then the roots of the characteristic equation corresponding to equation (45) will aways be complex, having

moduli equal to 1*.

The theorem gives only the sufficient conditions. If p is a positive constant the roots of the characteristic equation $\chi(x) = 0$, corresponding to the real period ω, will have moduli equal to unity. But if p is a positive periodic function, it is possible to find examples in which the characteristic equation has real roots, one of which one is greater than one (in absolute magnitude) and the other less than one.

The study of stability in the critical cases where the characteristic equation $\chi(x) = 0$ has one zero root or two imaginary roots with moduli equal to unity is similar to the investigation for equations with constant coefficients. These cases were considered by Lyapunov in sections 56-64 of his work "The general problem of stability of motion".

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* Further studies by Lyapunov connected with equation (45) were published in[2].

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